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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Why Not?

As we have before editorially pointed out, a strange curiosity in current educational practice is the coexistence in American schools of two antagonistic trends—one definitely headed for a diminished quantity of mathematical course material, the other a mounting demand for more mathematics to meet the needs of expanding and multiplying sciences.

We would be fortunate if the "curiosity" were merely academic. But it is not. Real elements of the near-tragic are there. This is true, not so much because of the implied danger to school mathematics through the slow crumbling of its secondary foundations, but rather because this crumbling must inevitably cripple the power of the sciences to serve humanity and increase the sum of human happiness.

Never before today have analysis, formulae, and processes of mathematics been so widely commandeered for the cause of science. It is equally true that never until now have the organized beginnings of school mathematics been so lightly regarded by administrations, so amazingly discounted by curriculum makers.

The seriousness of the anomaly is not lessened by our knowledge that its existence is traceable to fundamental differences between aims of the secondary and those of the higher levels of education.

Nor are the facts of this demand for more mathematics at the higher levels altered in the least by academic disputes concerning whether more or less of it should be offered at the secondary levels.

If the problem is to be solved quickly and effectively, a path beset with the least number of controversial difficulties—one already being traveled by some institutions—is open to us, namely: Let college curriculum makers be authorized to embody in the college course content all the mathematics heretofore assumed to be given by the accredited high school. Under the scheme, college mathematics would begin essentially with the elementary algebra and the plane (and solid) geometry, a reasonable amount of college credit being allowed for the same. The entering college freshman could waive the requirement to take these courses only by passing an examination in them, said examination being set for him by a member of the college mathematics staff. College credit need not in this case be allowed.

S. T. SANDERS.

The First Printed Armenian Treatise on Arithmetic and Algebra*

By ALLEN A. SHAW
University of Arizona

This is the first volume of the very valuable work of Father Lucas Derderiantz,† of the celebrated Mekhitarist order in Vienna. The whole work was published in 1843, Vienna, in three volumes,—Algebra, Geometry and Trigonometry. The general title is "*Elementary Mathematics*" and covers over 960 pages. While the text is called *elementary*, still a good deal of the material is of advanced character and a few topics are of unusual nature and worth introducing into our modern text books of similar contents. The work is in ancient Armenian and the symbols used are in Armenian script and not in Latin. It has educated generations of Armenian students in Venice, Vienna, Turkey, Armenia and Russia.

In the preface, the author says: "As this is the *first* text in Armenian and there are few teachers of mathematics, we have found it necessary to give lengthy presentations of the rules and explanations for the benefit of those who wish to study the text without a teacher, although we confess that it is very hard to accomplish this task without a teacher." The author further warns the reader to omit nothing in the text, all the laws of mathematics being so interconnected that without the mastery of each as it comes, progress is impossible.

At the ends of Vols. I and III the author gives a good glossary where Armenian mathematical terms are translated into German and French, thus enabling the reader to have access to the mathematical literature in those two languages. This is certainly highly commendable, as the modern student has frequent trouble in technical French and German and it is very desirable for the student of mathematics to become familiar with these terms as soon as possible.

Another point of interest about this text is that immediately after the preface, the author gives a concise but careful account of the landmarks of the history of mathematics, starting from the Egyptian period and finishing with the history of mathematics of the eighteenth

*Read before the American Association for the Advancement of Science at Lubbock, Texas, May 1, 1934.

†For a brief biography of Father Lucas, see NATIONAL MATHEMATICS MAGAZINE, Vol. 10, No. 8, p. 1.

century, thus enabling the reader to see the genesis and development of this exact science.

The first volume is divided into two parts: Part I is entirely arithmetic (pp. 1-36) and deals with our common numeral system, the four fundamental operations and denominate numbers. On p. 32 is an interesting and very valuable table of measures of weights, lengths, time and money—useful for any one interested in the study of Ananiah Shiragooni's work on the same subject. In his table of time, the author mentions and uses "thirds" (60 thirds makes one second) in his problems. The student of the history of mathematics will recall that great interest was attached to sexagesimal fractions (Babylonian in origin) by the Greeks in astronomical calculations, and we find them in full operation in Ptolemy and later writers.

Part II, which covers the rest of the text (pp. 37-450), is algebra, elementary and advanced, but arithmetic is not abandoned. While fully discussing various algebraic subjects, Father Lucas constantly introduces and develops arithmetical topics, such as rules of divisibility of numbers, highest common factors, least common multiples, vulgar and decimal fractions including the theory of recurring decimals, powers and roots, ratio and proportion, logarithms, etc. The reader feels that, while the author treats *algebra* quite fully, he makes it a handmaid to *arithmetic*. This is certainly a commendable point and should appeal to our professors of education who constantly condemn and discourage the teaching of algebra in our secondary schools as being an abstract science. Father Lucas has proved, beyond all doubt, the contrary.

Before considering the list of the author's purely algebraic topics, we wish to dwell on one or two of the valuable points in his arithmetic. One of these is his detailed and careful presentation of the rules of divisibility of numbers. I wish to emphasize this subject here because it is very sadly neglected in our text books on arithmetic and on this account the inability of the average student to handle his integers and fractions is simply deplorable, and it is high time that our arithmetic teachers paid attention to this very vital problem. Any student or teacher knows that our present arithmetic text books do not give rules of divisibility by any number except 2, 5 and 10. They invariably omit divisibility of a number by 3, 4, 6, 7, 8, 9, 11, 12 and 13, all of which are just as easy to teach, and equally important. Now, not only are all these fully explained and illustrated by Father Lucas, but the proofs of the rules are also very carefully given. The following is a summary of his rules of divisibility:

A number is divisible by:

2 if the last figure is even or 0;
3 if the sum of the digits is so divisible;
4 if the number formed by the last two digits is so divisible, or if it ends in two cyphers;
5 if the last figure is 5 or 0;
6 if it is divisible by 2, and also by 3;
7 (no simple rule, see later);
8 if the number formed by the last three digits is so divisible, or if it ends in three cyphers;
9 if the sum of the digits is so divisible;
10 if it ends in 0;
11 if the difference between the sums of the alternate digits, starting from the right, be 0 or a multiple of 11;
12 if it is divisible by 4, and also by 3;
13 (no simple rule, see later).

The rule for divisibility by 7 and by 13 (as well as an *additional* rule for divisibility by 11) is as follows:

Let N be any number in scale 10, then we have

$$N=(a+10b+100c)+(1000d+10000e+100000f)$$
$$+(1000000g+10000000h+100000000i)\ldots$$

where a, b, c, etc. are each less than 10.

Suppose we put α for the expression in the first bracket, 1000β for the expression in the second bracket, $1000^2\gamma$ for the expression in the third bracket, etc. We have:

$$N\div 7=(\alpha+1001\beta-\beta+1000000\gamma+1000000001\delta-\delta+\text{etc.})\div 7$$
$$=\alpha/7+143\beta-\beta/7+142857\ \gamma+\gamma/7+142857143\ \delta-\delta/7$$
$$+142857142857\ \epsilon+\text{etc.}=M(7)+1/7(\alpha-\beta+\gamma-\delta+\epsilon-\zeta+\text{etc.})$$

where $M(7)$ stands for a multiple of 7.

Hence if the expression in the last bracket is divisible by 7, N is so divisible.

Here the author leaves the rest of the proof (i. e. divisiblity by 11 and 13) as an exercise for the student. We shall, however, complete the proof and give a few illustrative examples, two of which are from the work of Father Lucas.

$$N \div 11 = (\alpha + 1001\beta - \beta + 1000000\gamma + 1000000001\delta - \delta + \text{etc.}) \div 11$$
$$= \alpha/11 + 91\beta - \beta/11 + 90909\gamma + \gamma/11 + 90909091\delta - \delta/11 + \text{etc.}$$
$$= M(11) + 1/11(\alpha - \beta + \gamma - \delta + \epsilon - \text{etc.}).$$

Hence if the expression in the last bracket is divisible by 11, N is so divisible.

Again

$$N \div 13 = (\alpha + 1001\beta - \beta + 1000000\gamma + 1000000001\delta - \delta + \text{etc.}) \div 13$$
$$= \alpha/13 + 77\beta - \beta/13 + 76923\gamma + \gamma/13 + 76923077\delta - \delta/13 + \text{etc.}$$
$$= M(13) + (\alpha - \beta + \gamma - \delta + \epsilon - \text{etc.})1/13.$$

Hence, if the expression in the last bracket is divisible by 13, N is so divisible.

Hence we have the *rule*:

Divide the number N into groups of three figures from the right (α, β, γ, *etc. as above*); *treat these groups as digits were treated in testing for the divisibility by* 11, *adding the groups alternately starting from the right and then subtracting the remaining groups*; *the number N is divisible by* 7, 11 *or* 13 *if the number left by this process is so divisible.* For example:

$$N = 847{,}963{,}207.$$

Then we have $207 - 963 + 847 = 91$.

Since 91 is divisible by 7 and 13 but not by 11, then N is divisible by 7 and 13 but not by 11.

The last period on the left may, of course, have fewer than three digits. For example:

$$N = 3{,}649{,}580{,}932{,}649{,}512{,}036{,}751{,}647{,}905$$

36	751	
649	512	
580	932	
3	649	
1915	3749	sum of odd-place groups
	1915	sum of even-place groups
	$1834 \div 7 = 262$	

Since 1843 is divisible by 7, then N is so divisible.

Again, $N=1{,}695{,}428{,}100{,}923{,}647$

$$\begin{array}{r|ll}
428 & 100 & \\
1 & 695 & \\
\hline
1352 & 1469 & \text{sum of odd-place groups} \\
 & 1352 & \text{sum of even-place groups} \\
\cline{2-2}
 & 117 \div 13 = 9. &
\end{array}$$

Since 117 is divisible by 13, but not by 7 and 11, then N is divisible by 13, but not by 7 or 11.

It is to be understood that the sum of the alternating sets is *algebraic*, i. e. the difference may be a ***negative*** number, e. g. in testing the number 2,684,623,571,273 by the above rule we find the difference as $-357 \div 7 = -51$; hence this number is divisible by 7.

The above theorem is more elegantly proved by means of congruences:

Let $\quad N = a + 10b + 10^2c + 10^3d + 10^4e + 10^5f + 10^6g + 10^7h + 10^8i + \ldots$

Since $\quad a \equiv a (\text{mod } 1001),\ 10 \equiv 10,\ 10^2 \equiv 10^2,\ 10^3 \equiv -1,\ 10^4 \equiv -10,$

$$10^6 \equiv -10^3 \equiv -(-1) \equiv 1,\ 10^7 \equiv 10,\ 10^8 \equiv 10^2, \text{ etc.}$$

$$\therefore N \equiv (a + 10b + 10^2c) - (d + 10e + 10^2f) + (g + 10h + 10^2i) - \text{etc.}$$

and the theorem follows.

The remaining chapters on arithmetic are equally interesting and well explained and illustrated by the author in his usual lucid style. The section on Compound Proportion and Proportional division is particularly well done and illustrated by a large number of worked examples of varying types.

The *algebraic* topics consist, among others, of the four fundamental operations: fractions; continued fractions; powers and roots—very fully treated (pp. 164-242); ratio and proportion, including compound proportion and proportional division; logarithms*; simultaneous equations with two or more unknowns, with theoretic explanations for solving n equations with n unknowns (pp. 372-3); quadratic equations; simple equations of higher degrees of the form $x^n = a$, for n odd or even, also equations of the type $x^{2n} + ax^n + b = 0$; indeterminate

*The book has no tables of logarithms. The author uses seven-place tables to secure accurate results. On pp. 304-5 is given a brief but interesting account of the history of logarithms and the logarithmic tables used before, and in, his time.

equations; arithmetical progressions, geometrical progressions and polygonal numbers.

To give the reader some idea of the *Algebra* of Father Lucas, we will present here his treatment of the polygonal numbers in modern symbols with a few additional comments. I chose to present this topic from his algebra not because it is the most important, but because it is the most neglected topic in our college algebras. Modern college algebras omit this topic because it is considered obsolete, or of less practical value, and this is a misfortune. The students of the history of mathematics will recall that polygonal numbers occupied the attention of Pythagoras and most of the early Greek mathematicians down to the time of Pascal. These numbers have their applications in modern mathematical analysis. See, for example the very important memoir on "*The Differential Equations of the Elliptic Cylinder*" by J. H. MacDonald in the "*Transactions of the American Mathematical Society*", Vol. 29 (1927), p. 651 where the author says: "...it may be noted that the number of terms of a given order in D_n is *figurate*. If $f_{n,r}$ denotes the nth figurate number of the rth order, the number of terms of the order $2p$ in D_n is $f_{2n+1-2p,\,p+1}$"

$$\text{where } D_n = \begin{vmatrix} 1 & X_1 & 0 & 0 & \cdot & \cdot & \cdot \\ X_2 & 1 & X_2 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & X_n & 1 \end{vmatrix} ."$$

Now these words of this important memoir would be perfectly meaningless if the reader had not studied polygonal and figurate numbers in his advanced algebra.

The following is the treatment of polygonal (Father Lucas calls them figurate) numbers.

Polygonal numbers are such a series of numbers that if in the place of each number we put the same number of dots symmetrically arranged, we form a *regular* polygon, such as an equilateral triangle, a square, a regular pentagon and so on.

If from the series of natural numbers, 1, 2, 3, 4, 5, 6, 7,..., n, where the common difference d is unity, we form another series by taking the first term and the successive sums of the first two, first three, first four, etc., terms, we obtain the series 1, 3, 6, 10, 15, 21 28,..., called *Triangular numbers*, in which the nth member is the sum of the first n terms of the natural numbers and is equal to

$$\frac{n(n+1)}{2}.$$

For example, the 7th triangular number is

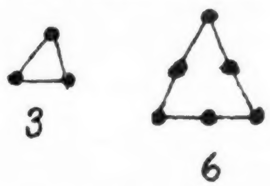
$$\frac{7\cdot 8}{2}=28.$$

In order to ascertain whether a given number A is triangular or not, we must put

$$\frac{n(n+1)}{2}=A, \quad \text{whence } n=\frac{-1+\sqrt{1+8A}}{2}.$$

For a triangular number the last equation should give us a positive integer. For example, let $A=703$, then by the above formula $n=37$, from which we know that 703 is the 37th triangular number. But if n is an irrational number or a fraction, then obviously A is not a triangular number.

EXAMPLES OF TRIANGULAR NUMBERS

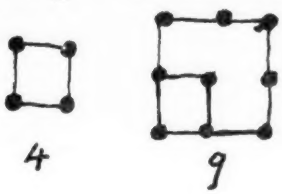


If we take the arithmetic series 1, 3, 5, 7, 9, 11,. . ., where $d=2$, and each member $=2n-1$, and as before we form another series by taking the first term, and the successive sums of the first two, first three, etc., terms we obtain the series 1, 4, 9, 16, 25, 36, 49, 81,. . ., called *Square numbers,* in which the nth member is the sum of the first n odd numbers, and is equal to

$$(1+2n-1)n/2=n^2.$$

It follows from this that the sum of the first n odd numbers, beginning with 1, is n^2.

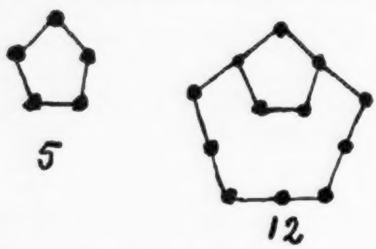
EXAMPLES OF SQUARE NUMBERS



If we take the arithmetic series 1, 4, 7, 10, 13, 16, 19,. . ., where $d=3$ and each member $=1+3(n-1)=3n-2$ and, as before, we form another series by taking the first term, and the successive sums of the first two, first three, etc., terms we obtain the *Pentagonal numbers* 1, 5, 12, 22, 35, 51, 70,. . ., where the nth member is the sum of the first n terms of the series 1, 4, 7,. . ., and is equal to

$$(1+3n-2)n/2=(3n-1)n/2.$$

EXAMPLES OF PENTAGONAL NUMBERS



The general case is given by the series

$$1,\quad 1+d,\quad 1+2d,\quad 1+3d,\quad \text{etc.}$$

where the common difference is d and the nth member is $1+(n-1)d$. The sum to n terms of this series is

$$[2+(n\ -1)d]n/2=\frac{n^2d-n(d-2)}{2}.$$

Now if in this general expression, whence polygonal numbers are derived, we put $d=1$, 2, 3, etc., we obtain triangular, square, pentagonal, etc. numbers. Hence if t indicates the number of angles of a polygon, then $d=t\ -2$. Therefore the above expression becomes

$$\frac{(t-2)n^2-(t-4)n}{2}$$

where n is the index of the number in the series of polygonal numbers. Hence if we wish to know the 7th hexagonal number, we must put $t=6$ and $n=7$ in the expression and the required number will be

$$\frac{4.7^2-2.7}{2}=\frac{2.7\,(2.7-1)}{2}=7\times 13=91.$$

The above discussion of polygonal numbers is almost a literal translation from Father Lucas' treatise and gives the reader a good idea of his style and mode of treatment of the various topics in his work.

Those who are acquainted with the ancient Armenian will find this volume a mine of information. It contains a large number of worked examples of different types—problems on business, physics, mechanics, as well as merely amusing problems.

As is usual with the European authors, the text contains no exercises for the student. They are to be supplied by the teacher himself—another point for commendation, which gives the teacher *individuality* and *freedom* to exercise his powers and select or construct his own problems according to the needs of his classes. Nowadays we are too slavishly adhering to the proofs and examples of the textbook, without having much freedom or opportunity for really constructive or creative work. In this respect some of us are mere parrots. We must remember that some of the European mathematicians, who produced valuable mathematical contributions and secured a permanent place in the history of mathematics, were teachers of secondary schools only, yet the system and the opportunity largely made them what they became.

Linear Derivative Inequalities and Differential Equations

By H. J. ETTLINGER
The University of Texas

We shall consider real functions of x defined on (0,1). A null set on (0,1) is a set which can be contained in a collection of segments whose total length is less than every pre-assigned positive number. A property holds "almost everywhere" on (0,1), if it holds for every x on (0,1) except a null set. We shall write "almost everywhere" briefly as a. e. The following auxiliary theorems concerning functions integrable in the sense of Lebesgue* will be used:

A. If $f(x)$ and $g(x)$ are (Lebesgue) L-integrable on (0,1) and $f(x) \leqq g(x)$ a. e. on (0,1), then

$$\int_{x_0}^{x_1} f(t) \leqq \int_{x_0}^{x_1} g(t),$$

where $0 \leqq x_0 \leqq x_1 \leqq 1$.

B. If $g(x)$ in auxiliary theorem A. is $|f(x)|$ then

$$\int_{x_0}^{x_1} f(t) \leqq \left|\int_{x_0}^{x_1} f(t)\right| \leqq \int_{x_0}^{x_1} |f(t)|.$$

We shall make use of the following properties of an absolutely continuous† function on (0,1):

If $v(x)$ is absolutely continuous on (0,1), 1) it has a derivative, $v'(x)$, a. e. on (0,1), 2) $v'(x)$ is L-integrable on (0,1), 3) $v(x) = \int_{x_0}^{x} v'(t) + v(x_0)$, where x_0 is a value of x on (0,1).

We will prove the following

Theorem. Hypothesis: 1) $P(x)$ *and* $Q(x)$ *are* L-*integrable on* (0,1), 2) $v(x)$ *is absolutely continuous on* (0,1), 3) *a. e. on* (0,1) $v(x)$ *satisfies the inequality*

$$v'(x) + P(x)v(x) \leqq Q(x); \tag{1}$$

*E. W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier Series, vol. 1, third edition, (1927) Cambridge University Press, chapter VII, pp. 562-674. L—integrable is here used in the sense of summable of some writers. In writing integral relations, the usual differential of the variable of integration will be omitted, and where there is no possibility of ambiguity, the variable of integration itself will be omitted.

†Hobson, l. c., pp. 587-597.

Conclusion:

$$v(x) \leqq e^{-\int_{x_0}^{x} P}\left[\int_{x_0}^{x} Q(t) e^{\int_{x_0}^{t} P} + v(x_0)\right], \tag{2}$$

where $0 \leqq x_0 \leqq x \leqq 1$. *If* $0 \leqq x \leqq x_0 \leqq 1$, *the inequality sign of* (2) *is reversed.*

Proof: Multiply (1) by the positive absolutely continuous function $e^{\int_{x_0}^{x} P}$. We obtain

$$v' e^{\int_{x_0}^{x} P} + P(x) v\, e^{\int_{x_0}^{x} P} \leqq Q(x) e^{\int_{x_0}^{x} P}. \tag{3}$$

The right side of (3) is L-integrable on (0,1), since the product of a function L-integrable on (0,1) by a function continuous on (0,1) is L-integrable on (0,1). Furthermore, the left side of (3) is the derivative a. e. of the absolutely continuous product, $v(x) e^{\int_{x_0}^{x} P}$. By the auxiliary theorem A we have

$$\left[v(t) e^{\int_{x_0}^{t} P}\right]_{x_0}^{x} \leqq \int_{x_0}^{x} Q(t) e^{\int_{x_0}^{x} P},\ x \geqq x_0,$$

or

$$v(x) e^{\int_{x_0}^{x} P} - v(x_0) \leqq \int_{x_0}^{x} Q(t) e^{\int_{x_0}^{t} P},\ x \geqq x_0. \tag{4}$$

If we add $v(x_0)$ to both sides and then divide by the positive function $e^{\int_{x_0}^{x} P}$, we have conclusion (2). The case $x \leqq x_0$ is equivalent to multiplying through by -1 and hence the inequality sign is reversed‡.

Corollary 1. If $Q(x) \equiv 0$ a. e. on (0,1) and $v(x_0) = 0$, then $v(x) \leqq 0$ on $(x_0,1)$. This follows at once from the vanishing of the right side of the bracket of (2).

Corollary 1^a. If, in addition, we have $v(x) \geqq 0$ on (0,1) and $v(x_0) = 0$, then $v(x) \equiv 0$ on $(x_0,1)$.

Corollary 1^b. If $y_1(x)$ and $y_2(x)$ are absolutely continuous on (0,1) and satisfy the differential equation (6) and the auxiliary condition (7),

$$y'(x) = f(x,y(x)) \text{ a. e. on } (0,1), \tag{6}$$

$$y(x_0) = y_0, \tag{7}$$

or the relation in integral form

$$y(x) = \int_{x_0}^{x} f(t,y(t)) + y_0, \tag{8}$$

‡For the case of equality, relation (2) is the solution of the general, linear, first order, non-homogeneous differential equation.

where 1) $f(x,y)$ is defined in the rectangular strip, R, defined by x on (0,1), and all values of y;

2) $$\int_{x_0}^{x} |f(t,y_1(t)) - f(t,y_2(t))| \leqq \int_{x_0}^{x} g(t)|y_1(t) - y_2(t)|,$$

$g(x)$ non-negative on (0,1) a. e., and L-integrable on (0,1), 3) y_0 is a constant, and x_0 is on (0,1), then $y_1(x) \equiv y_2(x)$ on (0,1).

To prove corollary 1^b identify $v(x)$ as $|y_1(x) - y_2(x)|$ in corollary 1^a with $P \equiv -g$. By this corollary, we have $v(x) \equiv 0$ on $(x_0,1)$. If $x_0 = 0$ the corollary is proved; if $x_0 > 0$, we have from (6) that

$$v'(x) \leqq g(x)\, v(x) \text{ a. e. on } (0,1)$$

or by Auxiliary theorem A we have

(9) $$v(x) \geqq \int_{x_0}^{x} gv \text{ on } x \leqq x_0.$$

But substituting y_1 and y_2 into (8), subtracting and taking absolute values on both sides we have

$$|y_1(x) - y_2(x) \leqq \int_{x_0}^{x} |f(t,y_1(t)) - f(t,y_2(t))|$$

or

(10) $$v(x) \leqq \int_{x_0}^{x} gv$$

for every x on (0,1). From (9) and (10) we have

$$v(x) = \int_{x_0}^{x} gv$$

on $(0,x_0)$, or taking derivatives on both sides

(11) $$v'(x) = gv \text{ a. e. on } (0,x_0).$$

From the case of equality in relation (2) we obtain

$$v(x) \equiv 0 \text{ on } (0,x_0),$$

so that $v(x) \equiv 0$ on (0,1). Hence $y_1(x) \equiv y_2(x)$ on (0,1). This is the well known result of "uniqueness" for the case of an absolutely continuous solution of (8).

Corollary 2. If $v(x_0) = 0$, then $v(x) \leqq e^{-\int_{x_0}^{x} P} \int_{x_0}^{x} Q(t) e^{\int_{x_0}^{t} P}$ for $x \geqq x_0$.

Corollary 2^a. If in addition, $Q(x) \equiv k$, where k is a constant, then

$$v(x) \leqq k e^{-\int_{x_0}^{x} P} \int_{x_0}^{x} e^{\int_{x_0}^{t} P}.$$

Corollary 2^b^. If in addition, $O \leqq P \leqq M$, a. e. on (0,1) then

$$v(x) \leqq (k/M)e^{M(x-x_0)}.$$

Corollary 3. If $P(x) \leqq 0$ a. e. on (0,1), then

$$v(x) \leqq C\left[\int_{x_0}^{x} Q + v(x_0)\right], \tag{12}$$

where $C = e^{-\int_0^1 P}$ and $x \geqq x_0$. This follows from the fact that since P is non-positive a. e. on (0,1), the exponential factor inside the bracket in (2) can be replaced by 1, and the outside exponential factor can be replaced by C. The inequality is thus strengthened and we obtain (12). Note. If $x \leqq x_0$, we replace the exponential factor inside the bracket by C and the outside exponential factor by 1. Now choose $\bar{C}$ as the larger of the two numbers, C and 1, then inequality (12) holds for every x on (0,1).

Corollary 4. If $y(x)$ and $Y(x)$ are absolutely continuous on (0,1) and satisfy the relations

$$y(x) = \int_{x_0}^{x} f(t,y(t)) + y_0$$

$$Y(x) = \int_{x_0}^{x} F(t, Y(t)) + Y_0,$$

where 1) $f(x,y(x))$, $F(x, Y(x))$ are defined in the rectangular strip R: x on (0,1) and all values of y and Y,

$$2)\ |f(x, Y(x)) - f(x,y(x))| \leqq g(x)\,|Y - y|,$$

$g(x)$ non-negative on (0,1) and L-integrable on (0,1). Identify $v(x)$ as $|y(x) - Y(x)|$ in Corollary 3. Then

$$|v'| = |f(x,y(x)) - F(x, Y(x))|$$

$$\leqq |f(x, Y(x)) - f(x,y(x))| + |F(x, Y(x)) - f(x, Y(x))|$$

or $$v' \leqq g(x)\,v(x) + |F - f|,$$

and $v(x_0) = |y_0 - Y_0|$.

By Corollary 3 we have for every x on (0,1)

$$v(x) \leqq \bar{C}\left[\int_0^1 |F - f| + |y_0 - Y_0|\right].$$

For every $\epsilon > 0$, there is a positive $\delta_\epsilon = \epsilon/2\,\bar{C}$, such that if $|y_0 - Y_0| < \delta_\epsilon$ and $\int_0^1 |F - f| < \delta_\epsilon$, then $|y(x) - Y(x)| < \epsilon$ for every x on (0,1). This last property is commonly expressed in the form that the absolutely

continuous solution of (8) is a continuous functional of the coefficients f and y_0 of the equation.

Corollary 5. Let $y(x)$ represent an absolutely continuous vector on (0,1) in n dimensions, i. e., $y(x)=(y_1(x),\ldots,y_n(x))$, where $y_j(x)$ is for each j on $(1,n)$ absolutely continuous in x on (0,1). Let $|y|=\sqrt{yy}=\sqrt{y_jy_j}$. Then $|y|$ is absolutely continuous on (0,1) and

$$|y|'=\frac{yy'}{|y|}\leqq\frac{|y||y'|}{|y|},\ |y|\neq 0.$$

Hence we have

$$|y|'\leqq||y|'|\leqq|y'|.$$

Let $y(x)$ and $Y(x)$ be two absolutely continuous vectors on (0,1) which satisfy

$$y'=f(x,y),\ y(x_0)=y_0,$$

where 1) $f(x,y)$ is the vector $(f_j(x,y_1(x),\ldots,y_n(x))$ defined in R_n: x on on (0,1), all values of y_j; x_0 on (0,1), y_{0j} any set of constants, 2) $|f(x,y)-f(x,Y)|\leqq g(x)|y-Y|$, where $g(x)$ is a scalar function, non-negative a. e. on (0,1) and L-integrable on (0,1). We will prove that $y(x)\equiv Y(x)$ on (0,1).

Identify $v(x)$ as the scalar function of corollary 1^b, $v(x)\equiv|y(x)-Y(x)|$, then

$$v'\leqq|v'|=|y-Y|'\leqq|y'-Y'|$$
$$\leqq|f(x,y)-f(x,Y)|$$

or

$$v'\leqq g(x)v,\ v(x_0)=0.$$

Following corollary 1^b, we obtain as before

$$v(x)\equiv 0 \text{ on } (0,1) \text{ or } y(x)\equiv Y(x) \text{ on } (0,1).$$

Corollary 6. Let the conditions of corollary 4 above hold for $y(x)$ and $Y(x)$ absolutely continuous vectors on (0,1) and equations of the same form as above hold in R_n. By the method of corollary 4 we obtain the result that the vector $y(x)$, or each component, $y_j(x)$, is a continuous function of all the f_j components of the given system and the n numbers, $y_{01},\ldots,y_{0n}$, in the sense that for every $\epsilon>0$, there is a $\delta_\epsilon>0$, such that if

$$\int_0^1|f(x,y(x))-F(x,y(x))|<\delta_\epsilon \text{ and } |Y_0-y_0|<\delta_\epsilon,$$

then

$$|y(x)-Y(x)|<\epsilon \text{ on } (0,1).$$

Goethe's Attitude Toward Mathematics[1]

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An unprejudiced study of Goethe's ideas and views on the essence of scientific thought may lead us to the conviction that in the future it will not be an easy task for historians who deal with this particular subject to find adequate reasons showing why Goethe was understood so little and misunderstood so much in this connection, as was the case during his lifetime and is still to a large extent the case today. Thus *Leitmeier* writes, for instance, in an article on mathematics printed in the *Goethe-Handbuch* (published by I. Zeitler. Stuttgart, Vol. 2, 1917):

"Auf die Mathematik war Goethe sehr schlecht zu sprechen, und es ist ja an und für sich auch begreiflich, dass ein grosser Dichter zu der trockensten, aber auch exaktesten aller Wissenschaften kein Verhältnis finden kann; er macht des öfteren seiner Abneigung durch teilweise recht gewagte Behauptungen Luft. . . ."

["Goethe did not care for mathematics and in itself it is quite comprehensible that a great poet should see nothing in the driest and the most exacting of all sciences. Quite frequently he expresses his dislike by sometimes rather bold assertions."]

E. Zimmer writes in his book "*Umsturz im Weltbild der Physik*" (Munich, 1934) p. 21:

"Es tritt für uns deshalb das erkenntnistheoretische Problem auf: hat Goethes Forschung noch das wesentliche Kennzeichen echter Naturwissenschaft? Ist sie noch allgemein gültig? Entsteht durch Verknüpfung nur sinnlicher Elemente, wie Goethe es will, eine für alle gültige Erkenntnis der Aussenwelt? Kann man z. B. einen Massstab angeben, an dem man die Allgemeingültigkeit eines Urphaenomens erkennen kann? Oder können etwa verschiedene Forscher für dieselbe Gruppe von Erscheinungen auch verschiedene Urphaenomene finden? Diese Frage ist bisher noch nicht befriedigend beantwortet worden."

["And so the problem of epistemology arises: are any essential characteristics of genuine science left in Goethe's investigations? Are these investigations still valid in general? Does a cognition of the outer world, valid for all cases, arise through the combination of merely sensory elements, as Goethe insists? Can one for example give a meas-

[1] After a lecture delivered at the International Congress of Mathematicians in Oslo (July, 1936) and an article bearing the same title which appeared in the periodical "*Das Goetheanum*" 15th Vol. (1936), No. 29, p. 227 (Dornach, Switzerland)

uring rod by which the general validity in all cases of a prime phenomenon can be known? Or can, in a given case, different investigators find different primal phenomena for the same group of occurrences? This question has not yet been answered satisfactorily."]

The reason why we are quoting these words here will be seen later. Below we would like to render some of Goethe's most important statements on mathematics and add a few comments which will make them more comprehensible. Goethe made different kinds of statements, and unless we know what gave rise to them and when they were made, they can be thoroughly misunderstood. Many statements which apparently imply a rejection of mathematics were made only because Goethe was attacked or passed over in a silence which condemned him. If this is not borne in mind and if Goethe's statements are accepted without taking into consideration their real background, a completely distorted picture will be the result.

On several occasions Goethe remarked that it was not granted to him

"durch Zeichen und Zahlen, mit welchen sich höchst begabte Geister leicht verständigen, auf irgendeine Weise zu operieren."

["to operate in any way whatsoever by means of symbols and numbers, which highly gifted minds understand easily."]

(Goethe to Neumann, Jan. 18, 1826).

His encompassing and extensive scientific writings, particularly his *Theory of Colours* (this work alone consists of about 1300 pages), repeatedly induced him to investigate the problem of the relationship existing between mathematics and natural sciences. His diaries for instance, contain notes such as these:

"Fortgesetzte Gedanken über das Verhältnis der Mathematik zur Physik."

["Further thoughts concerning the relation of mathematics to physics."]

(July 19, 1819).

In this connection, to be sure, Goethe has not expressed his views coherently and in detail, but his aphorisms, his essays on natural sciences, his *Theory of Colours* and other writings contain a sufficient number of statements which can supply information to mathematicians in particular. His statements on mathematics may be viewed essentially under the following aspects: In the first place, statements on mathematical methods, then on mathematics itself, on the application of mathematics, on mathematicians as scientists, and finally on the didactic significance of mathematics as a basis for striving after knowledge. The following sentences belong to the last-named category:[2]

[2] From "*Sprüche in Prosa*". Quotations from this source will henceforth be marked "*Sp.*"

"Das Wort: es solle kein mit der Geometrie Unbekannter, der Geometrie Fremder, in die Schule des Philosophen treten, heisst nicht etwa: man solle ein Mathematiker sein, um ein Weltweiser zu werden.

"Geometrie ist hier in ihren ersten Elementen gedacht, wie sie uns im Euklid vorliegt, und wie wir sie einen jeden Anfänger beginnen lassen. Alsdann aber ist sie die vollkommenste Vorbereitung, ja Einleitung in die Philosophie.—

"Wenn der Knabe zu begreifen anfängt, dass einem sichtbaren Punkte ein unsichtbarer vorhergehen müsse, dass der nächste Weg zwischen zwei Punkten schon als Linie gedacht werde, ehe sie mit dem Bleistift aufs Papier gezogen, wird, so fühlt er einen gewissen Stolz, ein Behagen. Und nicht mit Unrecht, denn ihm ist die Quelle alles Denkens aufgeschlossen, Idee und Verwirklichtes, potentia et actu, ist ihm klar geworden; der Philosoph entdeckt ihm nichts Neues; dem Geometer war von seiner Seite der Grund alles Denkens aufgegangen."

["The statement that no one ignorant of geometry, no one a stranger to geometry, should enter the school of the philosopher, this statement does not mean necessarily that one ought be a mathematician in order to become a philosopher.

"Here geometry is thought of in its simplest elements as it is presented to us in Euclid and as we let every young student begin it. But that geometry is the most perfect preparation, I would even say, introduction to philosophy.

"When the boy begins to comprehend that an invisible point must precede a visible point, that the shortest distance between two points is thought as a line before it is drawn on the paper with the lead pencil, then he feels a certain pride, a certain satisfaction. Not without reason, for the source of all thinking is unlocked to him. Idea and realization, potentiality and act, have become clear to him. The philosopher discloses nothing new to him. The foundation of all thinking had become clear to him as a geometrician."]

In connection with Prof. K. D. M. Stahl's application (December, 1798) for a promotion Goethe writes:[3]

"Ich besitze von ihm einen kleinen Aufsatz, der eine Uebersicht sämtlicher mathematischen Wissenschaften enthält, und den ich als eine Probe seiner Methode allenfalls vorlegen kann.

"Da es eine wahre Wohltat für die Jugend ist, Mathematik soweit als möglich zu verbreiten und zu erleichtern, so möchte sein Gesuch und seine Person wohl Aufmerksamkeit verdienen."

["I have a little essay by him which contains a compendium of all mathematical sciences. I can submit this as an example, at least, of his method.

"Since it is a real benefaction to youth to spread the study of mathematics as much as possible and to lighten it, his application and his character may well deserve attention."]

These lines alone—in so far as there is any possibility of taking Goethe's words seriously—would suffice to contradict the above-quoted view published in a Goethe-Handbook(!):

"Auf die Mathematik war Goethe sehr schlecht zu sprechen,. . . ."

["Goethe had no use for mathematics,. . . ."]

The most important and, in my opinion, the least known assertions should be considered under the aspect of *"methods in mathematics."*[4]

[3] Sophien Edition IV 13, p. 363. (Secret Archives, Weimar). This passage was brought to my notice by *W. Lorey's* fine article: *"Goethe's Stellung zur Mathematik"*, in the collected works entitled *"Goethe als Seher und Erforscher der Natur"*, published by Joh. Walther, Halle, 1930. It contains above all information concerning mathematicians who entered into a relationship with Goethe.

[4] In this connection, compare above all *Rudolf Steiner's Introduction to Goethe's Scientific Works* (in the Kürschner Edition of the *Deutsche National-Literatur*). Also in bookform, published by the Philos.-Anthropos. Verlag, Dornach (Switzerland).

We mathematicians spin the net of our thoughts, connected with any one of our specialized fields, on the basis of a few fundamental propositions (axioms), which could also be designated as the "primal phenomena" relating to a particular field. The proof of a mathematical proposition, i. e., its orderly inclusion in the net of our thoughts, consists therein that a bridge must be thrown from what is expressed in the proposition to the primal phenomena, and this in accordance with certain rules. If we survey the whole demonstration, the mathematical proposition will finally assume a form which is essential for it, namely the form of primal phenomena which are linked up with one another. Of course, we must realize that a mathematical talent does not consist so much in being able to grasp these connections subsequently, but rather in being able to trace and discover them. The mathematician therefore makes use of thinking—in geometry, for instance—not so much in order to find out *"the thing in itself"* which is supposed to lie, as it were *behind* the points, straight lines, etc. of the problem in question, but in order to group the primal phenomena in such a way that the geometrical facts which were at first merely conjectured, will then appear indeed as something which is dependent on these primal phenomena. An insight into geometrical facts does not consist in discovering something *outside* the geometrical field, but rather in establishing a connection with certain other facts which also lie completely within the geometrical field. This very method, which is so obvious to a mathematician, is just what Goethe wished to apply—through an exact knowledge of the mathematical process of demonstration—also to other phenomena which do not manifest themselves merely within the limits of time and space. Many modern scientists think that this is impossible, for reasons which transcend the scope of this article. In any case, we reach the undoubtedly surprising (and for many people shocking) result that Goethe wished to apply the mathematical methods also to other fields of experience, and consequently for Goethe these mathematical methods were far more encompassing and important than for many of our mathematicians.

He thus treats the world of colours in accordance with mathematical *methods*. On the foundation of an enormous wealth of observations, in addition to tests carried out by him, he searched within the world of colours for phenomena which, after being grouped and linked up with one another in a corresponding way, were able to produce other "deduced" colour phenomena. The essential thing is that these *primal* phenomena represent facts *which lie completely within the field of the colours themselves,* just as a geometrical axiom expresses

something which lies completely within the geometrical field. The fact that this method has not been observed in Goethe, although he applied it quite consciously and expressed it clearly enough, is responsible, for instance, for Zimmer's strange criticism, a few sentences of which were quoted at the beginning of this article.

The essay entitled "*Der Versuch als Vermittler von Objekt und Subjekt*" (1793) contains the following words:

"Ich habe in den zwei ersten Stücken meiner *optischen Beiträge* eine solche Reihe von Versuchen aufzustellen versucht, die zunächst an einander grenzen und sich unmittelbar berühren, ja, wenn man sie alle genau kennt und übersieht, gleichsam nur Einen Versuch ausmachen, nur Eine Erfahrung unter den mannichfaltigsten Ansichten darstellen.

Eine solche Erfahrung, die aus mehreren anderen besteht, ist offenbar von einer *höhern Art*. Sie stellt die Formel vor, unter welcher unzählige einzelne Rechnungsexempel ausgedrückt werden. Auf solche Erfahrungen der höhern Art loszuarbeiten, halt' ich für höchste Pflicht des Naturforschers, und dahin weist uns das Exempel der vorzüglichsten Männer, die in diesem Fache gearbeitet haben.

Diese Bedächtlichkeit, nur das Nächste ans Nächste zu reihen, oder vielmehr das Nächste aus dem Nächsten zu folgern, haben wir von den Mathematikern zu lernen, und selbst da, wo wir uns keiner Rechnung bedienen, müssen wir immer so zu Werke gehen, als wenn wir dem strengsten Geometer Recheneschaft zu geben schuldig wären.

Denn eigentlich ist es die Mathematische Methode, welche wegen ihrer Bedächtlichkeit und Reinheit gleich jeden Sprung in der Assertion offenbart, und ihre Beweise sind eigentlich nur umständliche Ausführungen, dass dasjenige, was in Verbindung vorgebracht wird, schon in seinen einfachen Teilen und in seiner ganzen Folge da gewesen, in seinem ganzen Umfange übersehen und unter allen Bedingungen richtig und unumstösslich erfunden worden. Und so sind ihre Demonstrationen en immer mehr *Darlegungen, Rekapitulationen*, als *Argumente*. Da ich diesen Underschied hier mache, so sei es mir erlaubt, einen Rückblick zu thun.

Man sieht den grossen Unterschied zwischen einer mathematischen Demonstration, welche die ersten Elemente durch so viele Verbindungen durchführt, und zwischen dem Beweise, den ein kluger Redner aus Argumenten führen könnte. Argumente können ganz isolierte Verhältnisse enthalten, und dennoch durch Witz und Einbildungskraft auf Einen Punkt zusammengeführt und der Schein eines Rechts oder Unrechts, eines Wahren oder Falschen überraschend genug hervorgebracht werden. Eben so kann man, zu Gunsten einer Hypothese oder Theorie, die einzelnen Versuche gleich Argumenten zusammen stellen und einen Beweis führen, der mehr oder weniger blendet.

Wem es dagegen zu thun ist, mit sich selbst und andern redlich zu Werke zu gehen, der wird auf das sorgfältigste die einzelnen Versuche durcharbeiten und so die Erfahrungen der höheren Art auszubilden suchen. Diese lassen sich durch kurze und fassliche Sätze aussprechen, neben einander stellen, und wie sie nach und nach ausgebildet worden, können sie geordnet und in ein solches Verhältnis gebracht werden dass sie so gut als mathematische Sätze entweder einzeln oder zusammengenommen unerschütterlich stehen.

....Meine Absicht ist: alle Erfahrungen in diesem Fache zu sammeln, alle Versuche selbst anzustellen und sie durch ihre grösste Mannichfaltigkeit durchzuführen, wodurch sie denn auch leicht nachzumachen und nicht aus dem Gesichtskreise so vieler Menschen hinausgerückt sind. Sodann die Sätze, in welchen sich die Erfahrungen von der höheren Gattung aussprechen lassen, aufzustellen und abzuwatren, inwiefern sich auch diese unter ein höheres Prinzip rangieren....."

["In the first two parts of my optical contributions I have attempted to set up such a series of experiments, which are connected with each other and touch each other directly. In fact, when they are all known exactly and viewed as a whole they form, as it were, only one experiment and represent only one set of facts under the most varied aspects.

Such an experiment, which consists of several others, is clearly of a higher degree. It represents the formula by which countless individual arithmetic examples are expressed. To work forward to such experiments of the higher degree, I consider the

highest duty of the scientist. The example of the best men who have worked in this field points in this direction.

This cautious attitude of connecting only the next with the next, or rather of concluding the next from the next, we have to learn from mathematicians. And even then, when we make no use of mathematics, we must always set to work as if we were bound to give an accounting to the most rigorous geometricians.

For it is really the mathematical method which, on account of its deliberation and purity reveals immediately every jump in the assertion, and its proofs are really only circumstantial discussions showing that what is brought forward in a connection was already present in its simple parts and in (all of its) conclusions, has already been reviewed in its entire circumference and found correct and inexpugnable under all conditions. And thus its demonstrations are always rather presentations, recapitulations, than arguments. Since I am making this distinction here, let me be allowed a retrospect.

We see the great difference between a mathematical demonstration which carries the first elements through so many combinations, and the proof which a clever speaker can bring from arguments. Arguments can contain quite isolated conditions, and nevertheless can be brought to a single point through wit and imagination, and the appearance of a right or wrong, or of something true or false can be produced surprisingly enough. In the same way one can put together in favor of a hypothesis or theory the individual experiments like arguments and can bring a proof which dazzles more or less.

But on the other hand, he who is concerned with going to work honestly will work through most carefully the individual experiments and thus seek to develop the experiences of the higher nature. These can be expressed by short and tangible statements, can be placed next to one another, and, as they are gradually developed can be arranged and brought into such a condition that they stand as unshakable as mathematical statements, either individually or taken together.

.... My purpose is to collect all the experiences in this field, to set up all the experiments myself and to carry them through their greatest diversity whereby they will be easily repeated and not removed from the circle of vision of so many people. Then it is my purpose to set up the statements in which the experiences of the higher degree are expressed and to find out to what extent these too range themselves under a higher principle."]

The following lines are taken from the essay "*Erfahrung und Wissenschaft*" (dated Jan. 15, 1798). In it Goethe describes his procedure in a concise way. Take now, for instance, the case of a geometrician who tries to explain how, after long investigations, he finally arrives at the choice of his axioms, which contain everything else in a nutshell. After a careful examination, it will surprise us to find such close correspondence between these two examples.

"Bei meiner Naturbeobachtung und Betrachtung bin ich folgender Methode, so viel als möglich war, besonders in den letzten Zeiten treu geblieben.

Wenn ich die Konstanz und Konsequenz der Phänomene, bis auf einen gewissen Grad, erfahren habe, so ziehe ich daraus ein empirisches Gesetz und schreibe es den künftigen Erscheinungen vor. Passen Gesetz und Erscheinungen in der Folge völlig, so habe ich gewonnen, passen sie nicht ganz, so werde ich auf die Umstände der einzelnen Fälle aufmerksam gemacht und genötigt neue Bedingungen zu suchen, unter denen ich die widersprechenden Versuche reiner darstellen kann; zeigt sich aber manchmal, unter gleichen Umständen, ein Fall, der meinem Gesetze widerspricht, so sehe ich, dass ich mit der ganzen Arbeit vorrücken und mir einen höhern Standpunkt suchen muss.

Dieses wäre also, nach meiner Erfahrung, derjenige Punkt, wo der menschliche Geist sich den Gegenständen in ihrer Allgemeinheit am meisten nähern, sie zu sich heranbringen, sich mit ihnen (wie wir es sonst in der gemeinen Empirie thun) auf eine rationelle Weise gleichsam amalgamieren kann.

Was wir also von unserer Arbeit vorzuweisen hätten wäre:

1. *Das emirische Phänomen,*
   das jeder Mensch in der Natur gewahr wird, und das nachher

2. *zum wissenschaftlichen Phänomen*
   durch Versuche erhoben wird, indem man es unter andern Umständen und Bedingungen als es zuerst bekannt gewesen, und in einer mehr oder weniger glücklichen Folge darstellt.
3. *Das reine Phänomen*
   steht nun zuletzt als Resultat aller Erfahrungen und Versuche da. Es kann niemals isoliert sein, sondern es zeigt sich in einer stetigen Folge der Erscheinungen. Um es darzustellen bestimmt der menschliche Geist das empirisch Wankende, schliesst das Zufällige aus, sondert das Unreine, entwickeit das Verworrene, ja entdeckt das Unbekannte.

Hier wäre, wenn der Mensch sich zu bescheiden wüsste, vielleicht das letzte Ziel unserer Kräfte. Denn hier wird nicht nach Ursachen gefragt, sondern nach Bedingungen, unter welchen die Phänomene erscheinen; es wird ihre konsequente Folge, ihr ewiges Wiederkehren unter tausenderlei Umständen, ihre Einerleiheit und Veränderlichkeit angeschaut und angenommen, ihre Bestimmtheit anerkannt und durch den menschlichen Geist wieder bestimmt...."

["In my observation and consideration of nature I have been faithful as far as possible to the following method, especially in latter times.

When I have experienced constancy and consistency of phenomena up to a certain point, I conclude from that empirical law and prescribe it for future phenomena. If, later, law and phenomena fit perfectly then I have won; if they do not fit then my attention is drawn to the circumstances of the individual cases and I am forced to seek new conditions under which I can present the contradictory experiments more purely; but if under the same conditions a case which contradicts my law appears several times, then I see that I must move ahead with the entire work and seek for myself a higher standpoint.

This, according to my experience, would be the point where the human mind can most closely approach objects in their generality, can bring them close to it and can at the same time amalgamate itself with them (as we usually do in common empiricism) in a rational way, and so we have the following to show as the result of our studies:

1. *The empirical phenomenon*
   which every one perceives in nature and which afterwards is raised to
2. *The scientific phenomenon*
   through experiments, in that it is represented under other circumstances and conditions than it was at first known, and in a more or less happy series.
3. *The pure phenomenon*
   is now present, finally, as the result of all experiences and experiments. It can never be isolated, but shows itself in a continuous series of phenomena. In order to represent it, the human mind determines the empirically inconstant, excludes the accidental, separates the impure, develops the confused, yes, discovers the unknown.

Here, if man knew how to be modest, would perhaps be the last goal of our energies. For here there is no question of causes, but of the conditions under which phenomena manifest themselves; their consistent succession, their eternal returning under thousandfold circumstances, their uniformity and changeableness are observed and accepted; their definiteness is recognized and defined again by the human mind."]

Goethe often objected to the exclusive application of the principle of causality. In mathematics we have gradually learnt, particularly during the 19th century and in recent years, to answer questions relating to fundamental laws more in the light of "when" than of "why." A geometrical proposition, for instance, may be right if we take the hypotheses of Euclid's geometry, and may not be right if we take the hypotheses of another geometry. Consequently, instead of the causalities which are supposed to lie at the foundation of a

proposition, we rather seek the requirements which are needed for a fundamental law. Goethe, too, does not seek a world of causalities *lying behind* a world of facts, but in accordance with true mathematical methodics he tries to discover exactly how the processes contained in a world of facts are dependent on one another.

The following maxim (Sp.) which refers principally to the primal phenomena of colours, may, however, be applied just as well to mathematical axioms:

"Urphaenomen: Ideal, real, symbolisch, identisch. Ideal, als das letzte Erkennbare; real, als erkannt; symbolisch, weil es alle Fälle begreift; identisch, mit allen Fällen."

["Basic phenomenon: Ideal, real, symbolic, identical. Ideal, (in as much as) the ultimate recognizable; real, as recognized; symbolic, becauses it includes all cases; identical, with all cases.']

In his *Colour Theory* Goethe demands the same exactness used by a geometrician who is investigating his particular field. This also explains the following maxim (Sp.):

"Was ist an der Mathematik exakt als die Exaktheit? Und diese, ist sie nicht eine Folge des innern Wahrheitsgefühls?"

["What in mathematics is exact but its very exactness? And this—is it not a consequence of the inherent feeling for truth?"]

The judgment "to be exact" does not refer to the idea of a straight line or to the perception of the colour red, but to the way in which geometrical forms or colour phenomena dependent on certain primal phenomena which are themselves purely geometrical facts, or lying entirely within the world of colours, may be understood. The fact that the essential value of the colour red or of a triangle may differ, is of no importance as far as the methodical treatment of the subject in question is concerned. Even the question as to what may lie at the foundation of the essence of colours is of no importance in the methodical treatment of the world of colours as such. That certain facts pertaining to time and space (quantitative) which accompany colour phenomena may be explained with the aid of the theory of waves has no connection whatever with the question as to whether Goethe's method, which may be called in a wider sense a mathematical method, is justified or not. And now we may consider those sentences of Goethe's which are misunderstood so much and which have to some extent a somewhat harsh sound, namely those referring to the application of mathematics (mathematics in the stricter meaning of the word) to Nature. We should not interpret these words as if Goethe failed to acknowledge what lies at the foundation of modern technical science, namely the mathematical treatment of a complex of phenomena

from the quantitative aspect, when the fundamental phenomena which apply to it have been discovered. But Goethe objected to statements connected with laws which may eventually be contained in the purely qualitative sphere of phenomena, if such statements were merely based on mathematical arguments. This is evident, after all, particularly to a mathematician. For any result obtained with the aid of mathematical deductions contains exactly what the axioms of the field concerned can supply. And in so far as a complex of phenomena, viewed from a quantitative aspect, may reveal elements which satisfy the axioms lying at the foundation of any algebra or geometry, the quantitative statements relating to these phenomena may be deduced with the aid of this algebra or geometry. This, too, is connected with the fact which has often been expressed so clearly by Goethe, namely that a theory may be able to explain the quantitative conditions relating to a complex of phenomena, and these explanations may be correct to a large extent, whereas the thoughts lying at the foundation of a theory in connection with the qualitative aspect may be entirely false.

"Es ist belehrend, dass so viele tief-und scharfsinnige Männer nicht einsahen, wie eine Berechnung mit dem Phänomen vollkommen übereinstimmen kann und deswegen gleichwohl die das Phänomen erklärende Theorie falsch sein dürfte. In Praktischen gewahren wir's jeden Tag; doch in der Wissenschaft sollten auf der Höhe der Philosophie, auf der wir stehen und, obgleich mit einigem Schwanken, gegründet sind, dergleichen Verwechslungen nicht mehr vorkommen. (Paralipomena zur Chromatik, aeltere Einleitung)."

"Der Mathematiker ist angewiesen auf's Quantitative, auf alles, was sich durch Zahl und Mass bestimmen lässt, und also gewissermassen auf das äusserlich erkennbare Universum. Betrachten wir aber dieses, insofern uns Fähigkeit gegeben ist, mit vollem Geiste und aus allen Kräften, so erkennen wir, dass *Quantität* und *Qualität* als die zwei Pole des erscheinenden Daseins gelten müssen; daher denn auch der Mathematiker seine Formelsprache so hoch steigert, um, insofern es möglich, in der messbaren und zählbaren Welt die unmessbare mit zu begreifen. Nun erscheint ihm alles greifbar, fasslich und mechanisch, und er kommt in den Verdacht eines heimlichen Atheismus, indem er ja das Unmessbare, welches wir Gott nennen, zugleich mit zu erfassen glaubt und daher dessen besonders oder vorzügliches Dasein aufzugeben scheint." (Sp.)

"Die Mathematiker sind wunderliche Leute; durch das Grosse, was sie leisteten, haben sie sich zur Universalgilde aufgeworfen und wollen nichts anerkennen, als was in ihren Kreis passt, was ihr Organ behandeln kann. Einer der ersten Mathematiker sagte bei Gelegenheit, wo man ihm ein physisches Kapitel andringlich empfehlen wollte: "Aber lässt sich denn gar nichts auf den Kalkül reduzieren?" (Sp.)

"Als getrennt muss sich darstellen: Physik von Mathematik. Jene muss in einer entschiedenen Unabhängigkeit bestehen und mit allen liebenden, verehrenden, frommen Kräften in die Natur und das heilige Leben derselben einzudringen suchen, ganz unbekümmert, was die Mathematik von ihrer Seite leistet und thut. Diese muss sich dagegen unabhängig von allem Aeussern erklären, ihren eigenen grossen Geistesgang gehen, und sich selben reiner ausbilden, als es geschehen kann, wenn sie wie bisher sich mit dem Vorhandenen abgiebt und diesem etwas abzugewinnen oder anzupassen trachtet." (Sp.)

"Die mathematischen Formeln ausser ihrer Sphäre, d. h. dem Räumlichen, angewendet, sind völlig starr und leblos, und ein solches Verfahren höchst ungeschickt. Gleichwohl herrscht in der Welt der von den Mathematikern unterhaltene Wahn, dass in der Mathematik allein das Heil zu finden sei, da sie doch, wie jedes Organ, unzuläng-

lich gegen das All ist. Denn jedes Organ ist spezifisch und für das Spezifische." (nach Gesprach mit Goethe auf gereichnet von *Riemer* 14. Jan. 1807.)

"Falsche Vorstellung, dass man ein Phänomen durch Kalkül oder durch Worte abthun und beseitigen könne." (Sp.)

"Es steht also hier die Bemerkung wohl am rechten Platze, dass man zwar irgend ein durch Erfahrung ausgemitteltes allgemeines Naturgesetz linear-symbolisch ausdrücken und dabei gar wohl die Umstände, wodurch das zum Grunde Liegende Phänomen hervorgebracht wird, voraussetzen könne; dass man aber von solchen Figuren auf dem Papiere nicht gegen die Natur weiter operieren dürfe, das man bei Darstellung eines Phänomens, das bloss durch die bestimmtesten Bedingungen hervorgebracht wird, eben diese Bedingungen nicht ignorieren, verschweigen, beseitigen dürfe, sondern sich Mühe zu geben habe, diese gleichfalls im allgemeinen auszusprechen und symbolisch darzustellen. . . ." (Zur Farben lehre, Newton's Optik Nr. 299).

"Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes." (Sp.)

"Wie man der französischen Sprache niemals den Vorzug streitig machen wird, als ausgebildete Hof= und Weltsprache, sich immer mehr aus= und fortbildend, zu wirken,so wird es niemand einfallen, das Verdienst der Mathematiker gering zu schätzen, welches sie, in ihrer Sprache die wichtigsten Angelegenheiten verhandelnd, sich um die Welt erwerben, indem sie alles, was der Zahl und dem Mass im höchsten Sinne unterworfen ist, zu regeln, zu bestimmen und zu entscheiden wissen." (Sp.)

"Jeder Denkende, der seinen Kalender ansieht, nach seiner Uhr blickt, wird sich erinnern, wem er diese Wohltaten schuldig ist. Wenn man sie aber auch auf ehrfurchtsvolle Weise in Zeit und Raum gewähren lässt, so werden sie erkennen, dass wir etwas gewahr werden, was weit darüber hinaus geht, welches allen angehört, und ohne welches sie selbst weder thun noch wirken könnten: *Idee* und *Liebe*." (Sp.)

"Ich ehre die Mathematik als die erhabenste und nützlichste Wissenschaft, solange man sie da anwendet, wo sie am Platze ist; allein ich kann nicht loben, dass man sie bei Dingen misbrauchen will, die gar nicht in ihrem Bereich liegen und wo die edle Wissenschaft sogleich als Unsinn erscheint. Und als ob alles nur dann existierte, wenn es sich mathematisch beweisen lässt! Es wäre doch töricht, wenn jemand nicht an die Liebe seines Mädchens glauben wollte, weil sie ihm solche nicht mathematisch beweisen kann! Ihre Mitgift kann sie ihm mathematisch beweisen, aber nicht ihre Liebe. Haben doch auch die Mathematiker nicht die Metamorphose der Pflanze erfunden! Ich habe dieses ohne die Mathematik vollbracht, und die Mathematiker haben es müssen gelten lassen. Um die Phänomene der Farbenlehre zu begreifen, gehört weiter nichts als ein reines Anschauen und ein gesunder Kopf; allein beides ist leider seltener als man glauben sollte." (Gespräche mit Eckermann 20. Dez. 1826.)

["It is enlightening, that so many profound and ingenious men could not understand how a calculation can completely agree with the phenomenon, and that yet the theory explaining the phenomenon could be wrong. In the practical field we notice this every day; however, in science, the theoretical field, such confusions should not occur in view of the high development of the philosophy on which we are based, in spite of that philosophy's occasional instability."

"The mathematician is dependent on certain quantities, on everything which can be determined by number and dimension, and so, so to speak, on the externally recognizable universe. However, if we regard the latter, in so far as we are capable, with all of our mind and all our powers, then we realize, that quantity and quality must stand for the two opposite poles of evident existence. Therefore the mathematician has so intensely elevated his equation language in order as far as possible to include the immeasurable with the measurable and countable world. Now everything appears to him tangible, comprehensible and mechanical, and he is suspected of veiled atheism; since thereby he believes to comprehend the immeasurable, which we call God, and thus seems to negate this God's special or extraordinary existence."

"The mathematicians are peculiar people; through the great things that they produce they set themselves up as a universal guild, and they do not want to acknowledge anything except that which fits into their sphere, that which their faculties can deal with. One of the foremost mathematicians once said, apparently ironically when a physical topic was urgently recommended to him: 'But can nothing at all be reduced to a mathematical formula?' "

"Physics must be distinguished from mathematics. The former must exist in complete independence and seek to penetrate into nature and the holy life of nature with all its loving, worshipping and pious forces. The latter, however, must declare

itself independent of all external things, pursue its own great mental process, and develop itself more surely, than can happen if it concerns itself as hitherto with the real world and seeks to profit from it or to adjust itselı to it."

"The mathematical formulas used outside of their sphere, i. e., space, are completely rigid and lifeless, and such a procedure is highly unsuitable. Nevertheless, there prevails in the world the erroneous idea, fostered by the mathematicians, that salvation is only to be found in mathematics; still, it is, like every organ, inadequate with respect to the universe." For each organ is specific and for specific fields. (After conversation with Goethe set down by Riemer 14 January, 1807.)

"It is an erroneous concept, that one could eradicate and remove a phenomenon through calculations or words."

"The remark probably is appropriate at this time, that one could indeed express in lines and symbols any general natural law, discovered through experience, and thereby even presuppose the circumstances through which the basic phenomenon is created; but that one should not continue to operate against nature with such figures that in the representation of a phenomenon, which is created only through the most definite conditions, one should ignore, conceal, remove these very conditions but should endeavor to express them and represent them symbolically also in a general way." (*zur Farbenlehre, Newton's Optik., Nr.* 299.)

"The mathematicians are like the French: If one speaks to them, they translate it into their language, and then it is at once something entirely different."

"Just as one will never deny the first place to the French language in operating as a developed court and world language,—as such ever improving itself,—thus it will not occur to anyone to minimize the merit of the mathematicians, which they, dealing with most important things in their language, earn for themselves, in that they are able to regulate, determine and decide everything which is subject to number and dimension in its highest sense."

"Any thinking person who looks at his calendar, at his watch, will remember to to whom he is indebted for these benefits. Yet even if one respectfully lets them work in time and space, they will realize that we perceive something which by far transcends time and space, something which belongs to all and without which they themselves could neither act nor operate: the *idea* and *love*."

"I honor mathematics as the sublimest and most useful science, as long as one employs it wherever suitable; but I cannot praise it when one desires to misuse it in connection with things that do not at all lie within its sphere and where this noble science at once appears as nonsense, and as if everything existed only if it permits mathematical proof. Would it not be foolish if someone would not want to believe in the love of his sweetheart because she cannot prove her love mathematically! She can prove her dowry to him mathematically, but not her love. The mathematicians have certainly not invented the metamorphosis of the plant either. I have accomplished this without mathematics and the mathematicians had to acknowledge its validity. In order to comprehend the phenomena of the color theory, there is needed only an unbiased observation and a clear head, but both unfortunately are rarer than one would think."] (Gespräche mit Eckermann 20 Dez. 1826.)

Goethe opposes the view that a mode of contemplation based on measures and numbers should constitute the only possible attitude of man toward Nature, and in his *Theory of Colours* he proves that the mathematical *methods* are not merely limited to processes taking place within time and space, and anyone who advances the argument that "in reality" colour is a process taking place within time and space is unable to rise to the capacity of allowing the great wealth of the world of colours, in its direct manifestation, to speak for itself, or of investigating whether, apart from the attendant phenomena limited to time and space, colour as such may not after all contain also certain primal phenomena.

"Das Recht, die Natur in ihren einfachsten, geheimsten Ursprügen sowie in ihren offenbarsten, am höchsten auffallenden Schöpfungen auch ohne Mitwirkung der Mathe-

matik zu betrachten, zu erforschen, zu erfassen, musste ich mir, meine Anlagen und Verhältnisse zu Rate ziehend, gar früh schon anmassen. Für mich habe ich es mein Leben durch behauptet. Was ich dabei geleistet, liegt vor Augen; wie es andern frommt, wird sich ergeben.

"Ungern aber habe ich zu bemerken gehabt, dass man meinen Bestrebungen einen falschen Sinn untergeschoben hat. Ich hörte mich anklagen, als sei ich ein Widersacher, ein Feind der Mathematik überhaupt, die doch niemand höher shhätzen kann als ich da sie gerade das leistet, was mir zu bewirken völlig versagt worden. Hierüber möchte ich mich gern erklären und wähle dazu ein eignes Mittel, solches durch Wort und Vortrag anderer bedeutender und namhafter Männer zu thun."

["The right to consider, examine and grasp nature in its most simple and secret origins, as well as in its most apparent and obvious creations without the aid of mathematics, I assumed to myself at a very early age relying upon my disposition and environment. For myself I have maintained this all my life. What I have accomplished herein lies before all eyes; what advantage it will be to others, remains to be seen.

"I am compelled, however, to remark that a false interpretation has been deduced from my efforts. The charges were brought against me as though I were an opponent, a foe of mathematics in general, which branch however no one can estimate more highly than myself since it accomplishes precisely that which has been completely denied me to bring about. On this point, I should greatly like to make an explanation, and I choose to do this through means of my own, through the word and record of other significant and authoritative men of note."]

(Opening paragraphs of the essay "*Ueber Mathematik und deren Missbrauch sowie das periodische Vorwalten einzelner wissenschaftlicher Zweige.*" 1826.)

"Bewährt der Forscher der Natur ein frei und ruhig Schauen, so folge Messkunst seiner Spur mit Vorsicht und Vertrauen.

"Zwar mag in einem Menschenkind sich Beides auch vereinen, doch dass es zwei Gewerbe sind das lässt sich nicht verneinen."

["If the investigator of nature is to preserve free and placid contemplation, let the art of measurement accompany his track with caution and confidence.

To be sure, both may be united in a human being, but it is not be denied that there are two branches."]

(From a letter of April 19, 1810 to G. Sartorius von Waltershausen. Jahrb. of the Goethe Society, 15th Vol. 1929.)

"Es folgt eben gar nicht, dass der Jäger, der das Wild erlegt, auch zugleich der Koch sein müsse, der es zubereitet; zufälligerweise kann ein Koch mit auf die Jagd gehen und gut schiessen; er würde aber einen bösen Fehlschuss thun, wenn er behauptete, um gut zu schiessen müsse man Koch sein. So kommen mir die Mathematiker vor, die behaupten, dass man in physischen Dingen nichts sehen, nichts finden könne, ohne Mathematiker zu sein; da sie doch immer zufrieden sein könnten, wenn man ihnen etwas in die Küche bringt, das sie mit Formeln spicken und nach belieben zurichten können. (Sp.)

["It does not follow that the hunter who kills the game must at the same time be the cook too, who prepares the game; a cook may accidentally go along on the hunt and shoot well; he would however draw the wrong conclusion if he were to maintain that one must be a cook in order to shoot well. Thus do I regard the mathematicians who maintain that one cannot see or find anything in physical matters, without being a mathematician; since they might be satisfied with dressing with formulas and preparing satisfactorily as they desire whatever is brought into the kitchen."]

From the great work *Entwurf einer Farbenlehre* we quote the following paragraphs:

722. Man kann von dem Physiker, welcher die Naturlehre in ihrem ganzen Umfange behandeln will, verlangen, dass er Mathematiker sei. In den mittleren Zeiten war die Mathematik das vorzüglichste unter den Organen, durch welche man

sich der Geheimnisse der Natur zu bemächtigen hoffte und noch ist in gewissen Teilen der Naturlehre die Messkunst, wie billig, herrschend.

723. Der Verfasser kann sich keiner Kultur von dieser Seite rühmen und verweilt deshalb nur in den von der Messkunst unabhängigen Regionen, die sich in der neuern Zeit weit und breit aufgethan haben.

724. Wer bekennt nicht, dass die Mathematik als eins der herrlichsten menschlichen Organe der Physik von einer Seite sehr vieles genutzt. Dass sie aber durch falsche Anwendung ihrer Behandlungsweise dieser Wissenschaft gar manches geschadet, lässt sich auch nicht wohl leugnen, und man findet's hier und da notdürftig eingestanden.

727. Der Verfasser des Gegenwärteigen hat die Farbenlehre durchaus von der Mathematik entfernt zu halten gesucht, ob sich gleich gewisse Punkte deutlich genug ergeben, wo die Beihilfe der Messkunst wünschenswert sein würde. Wären die vorurteilsfreien Mathematiker, mit denen er umzugehen das Glück hatte und hat, nicht durch andre Geschäfte abgehalten gewesen, um mit ihm gemeine Sache machen zu können, so würde der Behandlung von dieser Seite einiges Verdienst nicht fehlen. Aber so mag denn auch dieser Mangel zum Vorteil gereichen, in dem es nunmehr des geistreichen Mathematikers Geschäft werden kann, selbst aufzusuchen, wo denn die Farbenlehre seiner Hilfe bedarf und wie er zur Vollendung dieses Teils der Naturwissenschaft das Seinige beitragen kann.

[From the great work *Sketch for a Law of Color*, we quote the following paragraph:

722. One should demand of the physicist who is to treat of the laws of natural science in its entire extent that he be a mathematician. In the intermediate period, mathematics was considered the best means by which to come into possession of the secrets of nature, and yet in certain phases of the study of nature the art of measurement is properly paramount.

723. The writer cannot boast of any acquirements in this field, and therefore lingers only in regions of the art of measurement which have opened up far and wide in modern times.

724. Who does not recognize that mathematics, as one of the most splendid human instruments of Physics, in one aspect has been of great service? But that it has done much harm to this science through an improper application of the method of its treatment cannot indeed be denied, and one finds admissions of it under pressure.

727. The present writer has throughout endeavored to separate the law of color from mathematics, although certain points are clearly raised where the assistance of the art of measurement would be desirable. If the unbiased mathematicians with whom he had and has the good fortune to assoicate had not been prevented by other business from identifying themselves thoroughly with the subject, the treatment from this angle would not lack considerable advancement. But then this need may also turn into an advantage, in that it can now become the business of the penetrating mathematician to seek out himself where the theory of color needs his help and how he can contribute his own share to the completion of this part of natural science.]

The following statements contained in "*Materialien zur Geschichte der Farbenlehre*" prove how deeply Goethe perceived the essence and nature of mathematics: namely that it is a purely ideal, and in a certain sense free creation of the human spirit, arising entirely out of man's inner being. In connection with *Roger Bacon* he writes:

"Die Schriften Bacons zeugen von grosser Ruhe und Besonnenheit. Er fühlte sehr tief den Kampf, den er mit der Natur mit der Ueberlieferung zu bestehen hat. Er wird gewahr, dass er die Kräfte und Mittel hiezu bei sich selbst suchen muss. Hier findet er die Mathematik als ein sicheres, aus seinem Innern hervorspringendes Werkzeug. Er operiert mit demselben gegen die Natur und gegen seine Vorgänger, sein Unternehmen glückt ihm und er überzeugt sich, dass Mathematik den Grund zu allem Wissenschaftlichen lege.

"Hat ihm jedoch dieses Organ bei allem Messbaren gehörige Dienste geleistet, so findet er bald bei seinem zarten Gefühle, dass es Regionen gebe, wo es nicht hinreicht. Er spricht sehr deutlich aus, dass sie in solchen Fällen als eine Art von Symbolik zu

brauchen sei; aber in der Ausführung selbst vermischt er den reellen Dienst, den sie ihm leistet, mit dem symbolischen; wenigstens knüpft er beide Arten so genau zusammen, dass er beiden denselben Grad von Ueberzeugung zuschreibt, obgleich sein Symbolisieren manchmal bloss auf ein Witzspiel hinausläuft. In diesem Wenigen sind alle seine Tugenden und alle seine Fehler begriffen.

"Ein grosser Teil dessen, was man gewöhnlich Aberglauben nennt, ist aus einer falschen Anwendung der Mathematik entstanden, deswegen ja auch der Name eines Mathematikers mit dem eines Wahnkünstlers und Astrologen gleichgalt. Man erinnere sich der Signatur der Dinge, der Chiromantie, der Punktierkunst, selbst des Höllenzwangs; alle dieses Unwesen nimmt seinen wüsten Schein von der klarsten aller Wissenschaften, seine Verworrenheit von der exaktesten. Man hat daher nichts für verderblicher zu halten, als dass man, wie in der neuern Zeit abermals geschieht, die Mathematik aus der Vernunft = und Verstandesregion, wo ihr Sitz ist, in die Region der Phantasie und Sinnlichkeit freventlich herüberzieht.

"In der Geschichte der Londoner Societät, 18. Jahrhundert, steht: 'Indem mancher mit Furcht und Abneigung sich gegen jede theoretische Behandlung erklärte, so behielt man ein grosses Zutrauen zu der Mathematik, deren methodische Sicherheit in Behandlung körperlicher Dinge ihr, selbst in den Augen der grössten Zweifler, eine gewisse Realität zu geben schien. Man konnte nicht leugnen, dass sie, besonders auf technische Probleme angewendet, vorzüglich nützlich war, und so liess man sie mit Ehrfurcht gelten, ohne zu ahnden dass, indem man sich vor dem Ideellen zu hüten suchte, man das Ideellste zugelassen und beibehalten hatte'."

["The writings of Bacon bear witness to a great composure and reflection. He felt very deeply the battle which he had to put up against nature and tradition. He was aware that he must look for the means and agencies for this in himself. Here he finds mathematics a trustworthy tool springing out of his own being. With it he operates against nature and against his predecessors, his undertaking is successful and he is onvinced that mathematics supplies the foundation for all sciences.

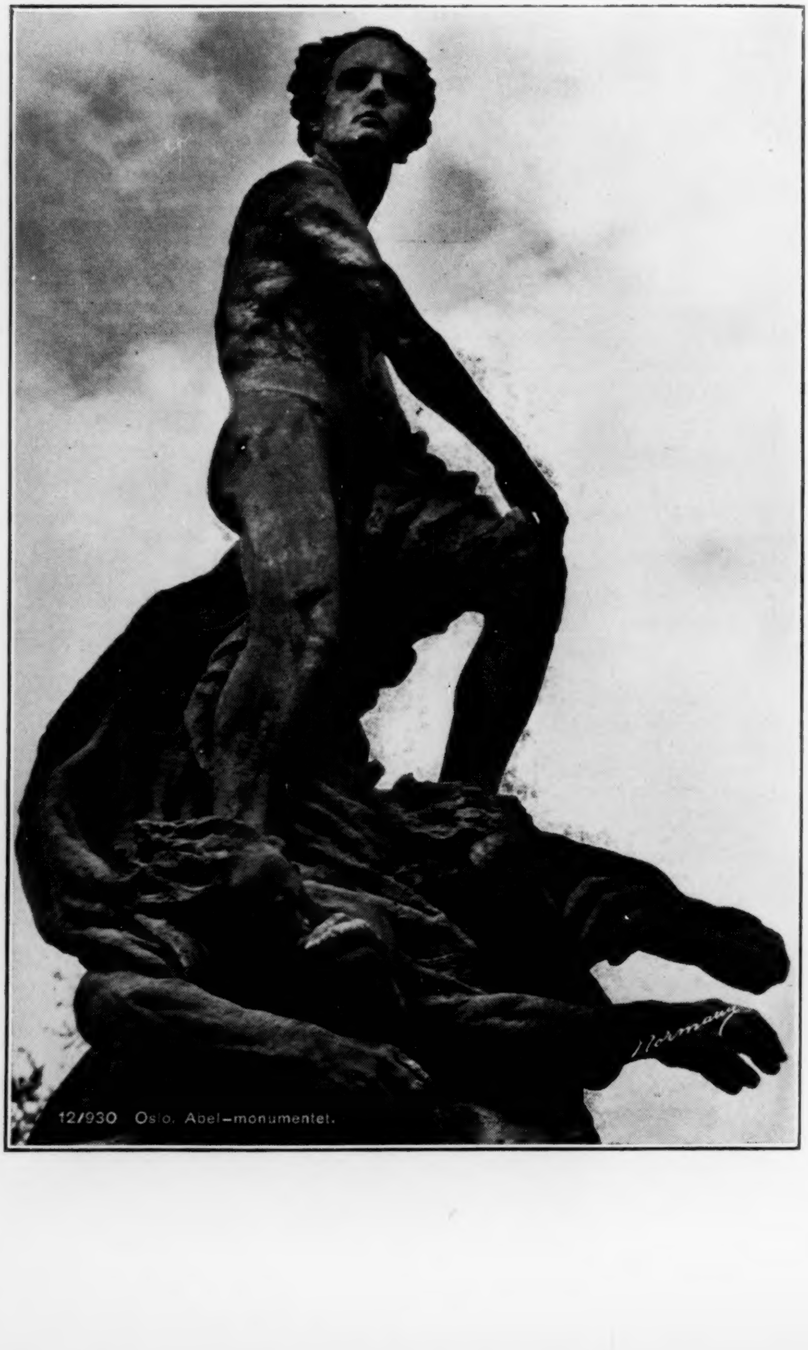
"But though this instrument has provided him competent services in all that pertains to measurement, he nevertheless discovers with sensitive feeling that there are regions which it does not touch. He mentions very pointedly that in such cases it is to be employed as a kind of symbolism; but in the execution he mingles the real service which it does for him with the symbolic; at least he connects both kinds so precisely that both are attributed the same degree of conviction, although his method of symbolizing often degenerates merely into a play of words. In this small matter often his virtues as well as his defects are displayed.

"A large portion of that which is commonly called superstition has arisen from a false application of mathematics, for which reason even the name of a mathematician was associated with that of a fortune-teller and an astrologist. One has only to recall the nomenclature of things such as chiromancy, geomancy, even the influence of evil spirits over men. All of this clap-trap takes its distorted light from the clearest, its confusion from the most exact of sciences. One should therefore regard nothing more harmful than criminally transferring mathematics from the region of reason and common sense, where its place is, to the region of phantasy and sensualism as often happens in modern times.

"In the history of the London Society of the 18th Century stands this: 'While many with fear and repugnance declared themselves against any theoretical treatment, yet one has a great confidence in mathematics whose methodical certainty in the treatment of inert objects seemed to provide it, even in the eyes of the greatest cynics, with a certain reality. One could not deny that it was extremely active especially when applied to technical problems with reverence, without divinity, that while a protection was sought against an ideal quantity, the most ideal quantity possible had been admitted and preserved'."]

In conclusion, a few sentences will be quoted which deal with the essence of mathematics, and in this connection with mathematicians:

"Die Mathematik ist, wie die Dialektik, ein Organ des inneren höheren Sinnes; in der Ausübung ist sie eine Kunst wie die Beredsamkeit. Für beide hat nichts Wert als die Form; der Gehalt ist ihnen gleichgiltig. Ob die Mathematik Pfennige oder Guineen berechne, die Retorik Wahres oder Falsches verteidige, ist beiden vollkommen gleich." (Sp.)



12/930 Oslo. Abel-monumentet.

"Hier aber kommt es nun auf die Natur des Menschen an, der ein solches Geschäft betreibt, eine solche Kunst ausübt. Ein durchgreifender Advokat in einer gerechten Sache, ein durchdringender Mathematiker vor dem Sternenhimmel erscheinen beide gleich gottähnlich." (Sp.)

"Der Mathematiker ist nur insofern vollkommen, als er ein vollkommener Mensch ist, als er das Schöne des Wahren in sich empfindet; dann erst wird er gründlich, durchsichtig, umsichtig, rein, klar, anmutig, ja elegant wirken. Das alles gehört dazu, um Lagrange ähnlich zu werden." (Sp.)

["Mathematics like dialectics is an organ of the higher mental processes; in practice it is an art like eloquence. For both, nothing has value but form; the content is immaterial. Whether mathematics reckons in pennies or guineas or rhetoric defends truth or falsehood, is absolutely immaterial to both.

"Here however it is a question of the nature of the man who practices such a study and trains in such an art. A brilliant lawyer in a just case, and a penetrating mathematician both appear equally divine in the sight of heaven.

"The mathematician is completely rounded only in so far as he is a completely rounded human being, and when he perceives the beauty of truth in himself; then only will he function thoroughly, penetratingly, broadly, purely, clearly, with elegance and charm. All that is necessary to become like Lagrange."]

We have not in any way exhausted the quotations in connection with Goethe's statements on mathematics and on what is related to mathematics; a more or less complete collection of quotations would be three times as long. But I think that the passages which were quoted here enable us to gather what is essential for our subject.

I wish to express my thanks to Mr. G. Waldo Dunnington for his kind invitation to write something about Goethe for this magazine. Goethe's mental stature is so great and so important for every human being that it may be necessary, even for us professional mathematicians, to realize clearly our connection with this lofty personality through our particular branch of knowledge. It is the scope and purpose of this article to contribute something to this task.

Vigeland's Monument to Abel in Oslo

By G. WALDO DUNNINGTON

In 1908 Norway honored the memory of Niels Henrik Abel by erecting a monument of him in the park in front of the Royal Castle in Oslo. Gustav Vigeland, the leading modern Norwegian sculptor, created this monument. The late Felix Klein in his *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (vol. 1, p. 108) compares Abel and Mozart, speaking in this connection of the beautiful monument to Mozart in Vienna. But of the Abel monument he writes:

"I cannot avoid referring on this occasion to the entirely different sort of monument which has been erected to Abel in Oslo and which must severely disappoint everyone who knows his nature. On a lofty steep block of granite a youthful athlete

of the Byronic type is striding upward, over two atrocious victims. In any case, if one can interpret the hero as a symbol of the human mind, then one questions in vain about the deeper meaning of this monstrosity. Are they conquered equations of the fifth degree or elliptic functions? The base of the monument bears in huge letters the inscription ABEL."*

Thorvald Holmboe, an engineer in Oslo and editor of a technical journal there, sent a letter in reply to Klein's criticism to Professor Wilhelm Lorey, to whom I am much indebted for calling my attention to it:

"Felix Klein's judgment probably is based on an ignorance of the independent artistic value of the Abel monument. The inspired Abel monument is not primarily to be interpreted as a portrait of the man Niels Henrik Abel; it is rather the homage of the artist to the genius of the mind, of thought—embodied in the greatest whom we have had. Therefore, also the simple vigorous inscription ABEL, purposely without any commentary.

"If Felix Klein had known the character of the Norwegian people better, that would certainly have been clear to him. The two allegorical figures, on whose backs the main figure is riding, are the embodiment of the genii ("guardian angels") of unconfined human thought, who carry the thought of a genius through the universe. The fact that these figures are represented somewhat grotesquely—and in execution they are quite naturalistic—agrees well with the idea of the severe and groping struggle of the human mind for truth.

"The masculine strength of the monument corresponds to the task before the artist much better than the trivial, mawkish figures which otherwise adorn the monuments of the large cities. A comparison with the Mozart monument in Vienna is irrelevant: those works of art originate in entirely different epochs and interpretations of art."

Albert Dresdner in a book on Swedish and Norwegian art since the Renaissance (1924) writes of the Abel monument:

"Vigeland is not so sure of himself in executing monuments. As a monument for the mathematician Abel he has created a splendid figure of a youth, winged with ideal buoyancy, but the relationship of this figure to those carrying it (which probably are to symbolize forces of nature) remains obscure."

On April 11, 1929, Vigeland's 60th birthday, Dresdner wrote in the *deutsch-nordisches Jahrbuch für Kulturaustausch und Volkskunde*, Jena, 1929, (p. 127) concerning the Abel monument:

"Vigeland's most excellent accomplishment in the field of monuments (as far as I know) is the monument for the mathematician Abel erected in 1908 on the Castle Hill in Oslo. On a mighty monolith rises abruptly a slender naked figure of a youth, borne by figures in swaying motion, who may symbolize mental powers or thoughts. A base for the figures is missing, the allegorical figures reach out over the edges of the monolith. The lower part of the group is full of unrest, rich in cross-cuttings, space-consuming openings and opposing motifs; but the body of the youth finally grasps and holds together the plastic form. The structure is naturalistic and illusionistic, the architectural principle is completely excluded. I couldn't say that this work appears to me thoroughly determined in every respect, but in the end one's impression is dominated by the nobility, power, and vibrancy of the youth's form rising in the open against the heavens, and it must not be overlooked that Vigeland has presented here the human being free and victorious through the mind."

*We are told that Vigeland felt that every Norwegian must know who Abel was, hence no more would be necessary.

∫	The Teacher's Department *Edited by* JOSEPH SEIDLIN	∫

The Place of Mathematics and Its Teaching in the Schools of this Country

By JOSEPH SEIDLIN

III.* SUMMARY AND ANALYSIS OF THE COMMENTS BY TEACHERS OF MATHEMATICS

And "comments" beget "comments" without end. It is less than certain that contents of courses or methods of teaching will be immediately affected by these comments of the first and second generation. But, without a doubt, as a group, we have "expressed ourselves." One of the most recent derivative comments contains the following prognosis: "*. . . In time to come these 132 comments will serve as a sott of bible. For as even the devil may plead his cause by an authentic quotation from the Bible, so any like-minded educationist may quote one or more of these comments to gain or prove his point.*"

It must be admitted that these comments have shown no landslide election of any one principle, position, or recommendation. But here and there we find a plurality opinion, appraisal, or judgment. There appears to be, also, greater unanimity in destructive criticism or evaluation than in any one constructive suggestion or proposed improvement.

For instance, it would be difficult to disguise the feeling of animosity, albeit impersonal, toward educationists as a group and schools of education as institutions. Though this expressed antagonism ranges from good-natured banter to "strong-language" raillery, it is in the main a clear-cut case of educational principle *vs.* educational principle rather than a group against a group: 124. . . . *Hope you get enough ammunition for a good shooting up of the Pedageese.* 96. . . . *Too many high school teachers in mathematics have been trained in teachers' socalled colleges.* 129. . . . *a much greater menace to mathematics than those intellectual dumps—the colleges of education. . . .*

*Cf. Teacher's Department, NATIONAL MATHEMATICS MAGAZINE, No. 1, Vol. XI, October, 1936.

Many, too many, comments of similar vein are stronger, non-printable in fact. It seems rather unfortunate, whatever the provocation, that educators, be they educationists or teachers of mathematics, should adopt the meanest accessories of political campaigning in order to prove or disprove, to establish or disestablish a principle of education. It may be argued that teachers of mathematics are merely replying in kind. Their backs have been broken by a lot of straws. That there is some truth in such assertions would not be denied by the more fair-minded educationists.

But, the innocent bystanders, the educands, have a right to inquire whether there are not more peaceful, if not more scientific, ways of settling controversial principles of education. Must the "doctors" disagree forever?

Even the less reasonable educationists no longer suggest that mathematics should be eliminated from the school curricula. Nor, on the other hand, do even the most conservative teachers of mathematics deny the possibility of improving the teaching and the content of mathematics courses. The more genuinely progressive educationists are apparently convinced that real mathematics courses well-taught are essential to the education of the "whole" child at any stage or level. A great many teachers of mathematics are becoming convinced that the teaching of mathematics, not only that of others but their own as well, needs to be improved; that the content of all courses needs to be re-evaluated and perhaps modified or completely changed.

Rarely does an educationist know enough mathematics to justify his assuming control of a course of study in mathematics or of his writing a textbook in mathematics. On the other hand, many a mathematician is educationally helpless in the matter of constructing a course of study or writing a textbook in the more elementary mathematics. Yet it would seem that comment 45 does not quite offer the solution: 45. ... *The content of courses in mathematics should be selected by mathematicians who are interested in education. In no case should a person who is an educator, interested in mathematics, be allowed to have a word to say*.... Much could be said for co-operation, here as elsewhere. Dogmatism and pedantry, certainly, are not the solution to a real problem.

Next in order of destructive criticism are the comments stressing the quality of the secondary school teachers. Some of these comments will bear reproduction: 7. ... *The trouble is not the subject so much as the teacher. We have high school teachers of mathematics in "P" who think it a waste of time to study mathematics*... 17. ... *In the high school of "K" bad teaching and bad learning are a consequence in large*

part of the ignorance of the teachers... 21. ... *It was true then, and it is also true now, that the average grade teacher and high school teacher knows little or nothing about mathematics...* 29. ... *I believe that of all subjects ours, as taught, is most pedantic. This, I believe, is due to the fact that, contrary to general opinion, mathematics is a refuge for an inferior type of teacher...* 37. ... *and my impression of the teaching profession, derived from the published utterances of their spokesmen, is that they are a bad lot. Therefore, ... in my opinion, the study of mathematics in high schools should be absolutely prohibited...* 64. ... *I recognize that most of the elementary teaching has been very inefficient because the teachers did not have sufficient preparation nor perspective...* 74. ... *The evidence is unanswerable that geometry as usually taught gives the average student neither the power to handle concrete geometric situations nor a transferable ability to draw valid conclusions from hypotheses...* 118. ... *Poor teaching gives high school mathematics its "black eye"...* 129. ... *The formal, manipulative, exercise-working teacher of mathematics is, in my opinion, a much greater menace to mathematics than...* 130. ... *The worst trouble, as I see it, is the poor quality of teaching that is generally done in high schools, preparatory schools, and other pre-college schools...*

The secondary school teachers, themselves, do not deny the prevalence of poor teaching in their ranks. They claim, however, that the inspiration for a good deal of that poor teaching is derived from at least as poor teaching of their teachers, many of whom were professors of mathematics. In other words, not all the poor teachers of high school mathematics are non-mathematics majors. So, perhaps, if it is a "poor thing", much of it, too much, is all our own. Comment 105 seems highly improbable, but there it is: "*I believe that we [college teachers] need not be especially concerned with problems of the high school. Let them [high school teachers] take care of their messy situation. We have our own.*"

The students we get in college mathematics are conditioned, in more ways than one, by these high school teachers. As teachers of these high school teachers what have we done to them to make them "*think it a waste of time to study mathematics*" (7). No, the whole problem of the teaching of high school mathematics is in a very real sense our problem. It may be that genuine co-operation between college teachers of mathematics and high school teachers of mathematics will prove to be one of the major means of improving, if not saving, "functional" education in our democracy.

Comments on the nature and contents of the courses as now taught or as visioned for the future are quite diversified and often,

too often, sadly contradictory: 24. ... *Genuine, solid courses in mathematics as provided by our better textbooks should be available—without dilution or sugar coating—...* 43. ... *So far as traditional content is concerned, I am of the opinion that it may be possible to develop an equivalent course of more significance than the present courses in algebra and plane geometry,...* 90. ... *I think a sound course in plane geometry is the best all-round course in our modern educational curriculum just as it was in Euclid's time and in all the two thousand years between...* 91. ... *I doubt the worth of geometry; unless treated in the analytic method...* 80. ... *The average high school student is not qualified to understand demonstrative geometry. In fact the superior college senior is not so qualified; neither is the average high school teacher of geometry...* 78. ... *No doubt the subject matter of mathematics can be and should be improved...* 35. ... *I believe that algebra and geometry at the high school level should be rigorous and should adhere closely to the classical content...* 8. ... *eventually there will be different types of mathematics courses in high schools and possibly in colleges...* 15. ... *I am strongly in favor of the old reliable plan of teaching hard mathematics in high school and college...*

There appears to be some unanimity of opinion that the present set-up of both high school and college courses is in need of change. There is very little agreement as to the nature of the change. A good many are resigned to wait for a miracle or a genius to bring about the right kind of revision: 79. ... *courses in mathematics should be revised so as to make them practical, real, and vital at the same time. As yet I do not know of any such courses...* 108. ... *I should like to see 3 hours or 6 hours of mathematics required, but totally different from any now taught in any college I know....* 113. ... *Sometimes a genius will appear who will frame the course that ought to be taught to all students in college...*

Last, but in a very real sense the most important controversial issue and certainly one that needs our most immediate consideration is: "*Shall mathematics be taught to the academically gifted few, or shall some sort* (?) *of mathematics be taught* (?) *to all whose presence is accounted for through the junior college level?*" It is not so much a choice between requiring or not requiring mathematics. It is even beside the point whether or not our decision will influence Boards of Education, or Superintendents, or faculties of universities. It is, rather, making a choice between a philosophy of education as implied in 24. ... *I would hate to have to teach mathematics to all students of a liberal arts college... Srongest conviction: Genuine, solid courses in mathematics as provided by our better textbooks should be available—without dilution or sugar coating—for the good students. I do not care what they do with*

the morons..., or that expressed in 6. ... *It is strictly up to mathematicians as a whole to make mathematics really and obviously worth while to all students. I would make drastic changes in content*... *When we as a whole make our work really worth while to all students we will not need to argue the question* [*Should mathematics be required*?] *But that is still a long story*...

How "long a story" need it be? Evidently, we are *not ready* to advise our curriculum builders when and if they ask for our advice. 73. ... *I am sufficiently convinced of the intrinsic worth of mathematical training even for the student who does not expect to go to college. But we must get rid of a lot of junk before we can urge this with a straight face*... What "junk"? Why do not we get rid of it? After all, sooner or later, we will have to "straighten out our faces." Is (92) a possible solution? 92. ... *For general use in high school and also in college a course about mathematics must be developed*... Is (95) utopian? 95. ... *I believe that every bona fide student in college* (*liberal arts*) *should have some knowledge of the calculus*... Is (99) sound? 99. ... *I believe the theory that students should do only those things they like to do has a great deal to do with the present unrest. We should try to get them to like to do those things that they should do*... If (37) is "correct", to what avail are our best efforts? 37. ... *My personal belief is that efforts to save mathematics are doomed to fail, until the present retrogression of the American people towards barbarism is stayed. This has been going on since 1900 and I see no signs of a reversal. The attack on mathematics is merely a small part of the war against excellence of any kind*...

Is it pure wishful thinking to assume that "enriched" (in the best sense) and better taught courses of mathematics permeating the curricula of our schools at all pre-professional levels may bring about a war to end the "war against excellence of any kind." In the final analysis, would not that be a grand "cardinal" objective in the educational scheme of a democracy?

It is rather difficult to epitomize the expression of opinion of so large and representative number of our colleagues. As a group we are quite uncertain about the value of what now passes for instruction in mathematics. As educators we are convinced that courses in mathematics, well-constructed and well-taught, are paramount, if not indispensable, in any reasonable educational scheme of an ultimately-enlightened democracy. To make well-built and well-taught courses in mathematics a reality, may require the finest cooperative efforts of mathematicians, teachers of mathematics, and educationists. Not many of these, barring a few incurable psychopaths, would disagree with the last sentence of comment 98: "Obviously, *to omit* is *not to solve* the problem of mathematics in curricula."

Textbooks, Reference Books or Story Books

By HARRIET F. MONTAGUE
University of Buffalo

It is often said that the college teacher has little sympathy with "education", i. e. with methods and theories of teaching. It might truly be said that the majority of college teachers outside the field of education do decry the work of the educationists in trying to improve the caliber of teaching in the schools. These "anti-educationists" feel that successful teaching is applied common sense and that a good teacher cannot be made despite all the theory about "education" to which a prospective teacher may be exposed. To them a teacher's main task is the dispensing of subject matter. Whether his method of doing so may be improved or not is beside the point.

This article is not written by such an educational "outsider." Although I may not be catalogued as an educationist, I feel quite deeply about problems of teaching, especially those concerned with the overlap of secondary and college teaching. This is not submitted, therefore, in a spirit of antagonistic criticism but in recognition of certain problems which do not exist and are very seldom discussed.

The question raised in the title has confronted me at increasingly frequent intervals since the overlap in junior college and secondary schools has grown more noticeable. The textbook for secondary and primary schools has been made the subject of investigation by professional organizations but has anyone considred the problem of the textbook in relation to the overlap? Such a question should be challenging.

Many colleges no longer have entrance requirements in elementary mathematics, intermediate or second year algebra in particular. As a result many students enter college courses without the training in mathematics prerequisite to those courses. A most common example is the pre-professional student who must take courses in sciences requiring knowledge of algebra, geometry, and trigonometry. If he has not obtained this knowledge in a secondary school, he must do so in the junior college. It becomes necessary for the mathematics departments to offer courses in algebra equivalent to secondary school intermediate algebra in subject matter. It has been the writer's task in recent years to select textbooks for use in these courses. A striking difference in college and secondary school texts is evident, a difference arising from fundamental differences in teaching objectives.

An examination of a modern secondary school textbook in mathematics shows a praiseworthy attempt to correlate mathematics to the world as a whole by means of illustrations, historical notes, and explanatory notes on scientific applications. The college text, on the other hand, is found to be concerned primarily with a presentation of subject matter without much attempt at correlation. The first book aims at securing the interest of the student in the subject—the second presupposes that interest. But should a textbook be expected to supply that interest? Is not that the task of an adequate teacher?

It is my opinion that one of the primary objects of a college education is the development in students of an ability to use books as reference books. Their textbooks should become for them old friends who will answer their questions quickly and easily. Our secondary school graduates are lamentably lacking in this ability. Textbooks seem to be of no use to them except as they fix their attention on certain pages where problems have been assigned. Confessions from both students and teachers confirm this opinion. Can it be that all the illustrations and explanatory notes hide the essential subject matter so that the student is discouraged in the use of his book as a reference book? The compact arrangement of the college text is certainly much more encouraging in the search for an answer to a perplexing phase of a problem.

There is a tendency now to do the same type of sugar-coating in the college texts. It has not gone nearly as far as in the high school texts, however. We appreciate the fact that on the whole textbooks of today are much superior to those of fifty years ago. But before the present tendency of sugar-coating goes too far it might be well to stop and consider what we have done. We try to lead our students to a realization of the existing knowledge in the world. Should we discourage them from seeking it by hiding it in textbooks which are not reference but story books?

Mathematical Notes

Edited by
L. J. ADAMS

Professor Alfred Hume, University of Mississippi, reports the following changes in the Department of Mathematics at that institution:

1. Prof. Robert Torrey, part-time Assistant Professor, after having served many years, retired at the end of last session.

2. Prof. T. P. Scott, Registrar of the University and part-time Assistant Professor of Mathematics, is now teaching in Howard College, Birmingham, Alabama.

3. Dr. Geo. W. Nicholson has been appointed Assistant Professor of Mathematics. He has held positions in the University of the South, Sewanee, Tennessee, Georgia Tech, and Brenau College before coming to the University of Mississippi. He had his Doctor's degree from the University of North Carolina.

4. Dr. A. B. Lewis has been appointed Associate Professor of Mathematics and Physics. Doctor Lewis has his Doctor's degree from Johns Hopkins University and has had about ten years' experience in the United States Bureau of Standards.

5. Prof. T. A. Bickerstaff, formerly full-time Assistant Professor of Mathematics, is now Registrar and part-time Assistant Professor of Mathematics.

Professor Philip Franklin, Massachusetts Institute of Technology, spent last year in research at the Institute for Advanced Study, Princeton.

Professor Norbert Wiener, Massachusetts Institute of Technology, spent the year as Research Professor of Mathematics at the Tsing Hua University, Peiping, China. On his way home he attended the International Congress of Mathematicians at Oslo and gave one of the invited addresses.

For his work on the problem of Plateau Professor Jesse Douglas, Massachusetts Institute of Technology, was awarded a Fields gold medal at the Congress in Oslo. Two of these medals were awarded for the best mathematical research done during the preceding four

years. Professor Douglas has also been invited to deliver the colloquium lectures of the American Mathematical Society at its summer meeting in 1937.

News items from the University of Michigan include the following: Professor H. C. Carver has been promoted from Associate Professor to Professor, Professor C. C. Craig, from Assistant Professor to Associate Professor, and Drs. Ben Dushnik and E. W. Miller from Instructors to Assistant Professors. In addition, Drs. Ralph Hull, Sumner B. Myers, and Merrill E. Shanks have been appointed Instructors in Mathematics.

Miss Lida B. May, Lubbock, has been appointed graduate assistant in mathematics at Texas Technological College as a result of the overcrowded conditions in the department. Miss May holds the master's degree from the University of Texas and is a member of the Mathematical Association of America and Pi Mu Epsilon.

"We have 200 more students enrolled in mathematic courses than ever before in the history of Texas Tech," states Dr. J. N. Michie, head professor of mathematics.

Dr. Oswald K. Sagen of the University of Minnesota has been appointed to an instructorship in mathematics at Iowa State College. Dr. Sagen secured his Ph.D. at the University of Chicago and subsequently studied at the Institute for Advanced Study.

Professor H. F. Blichfeldt, Stanford University, attended the International Congress of Mathematicians held in Oslo in July as one of the delegates of the United States Government.

Professor Harold M. Bacon, Stanford University, read a paper and led the subsequent discussion on the subject "*How Will the New Trends in Education Affect Junior College Mathematics*?" at the meeting of the mathematics section of the Northern California Association of Junior Colleges held in Sacramento in April.

The following changes have occurred at the Colorado School of Mines in the Department of Mathematics: Professor I. L. Habel has been promoted to Associate Professor, and Mr. Norman J. Castellan has been added to the Department staff as Instructor.

Miss Elizabeth Shelburne, Instructor of Mathematics, has been appointed Assistant Dean of Women at Texas Christian University. However, she will remain in the Department of Mathematics for part time teaching.

Mr. H. J. Jones has been appointed an Instructor of Mathematics at Texas Christian University. He has been teaching in the Public Schools at Wichita Falls, Texas.

Mr. Henry F. Schroeder has been appointed an Assistant Professor of Mathematics at Louisiana Polytechnic Institute. For several years Mr. Schroeder was Principal of the Training School of Louisiana Polytechnic Institute.

The National Council of Teachers of Mathematics announces its eleventh yearbook: *The Place of Mathematics in Modern Education.* The contents by chapters include:

1. *Criticisms of Mathematics, Pro and Con.* W. D. Reeve.
2. *Reorganization of Mathematics.* Wm. Betz.
3. *The Meaning of Mathematics.* E. T. Bell.
4. *The Contribution of Mathematics to Civilization.* David Eugene Smith.
5. *The Contribution of Mathematics to Education.* Sir Cyril Ashford.
6. *Mathematics in General Education.* W. Lietzmann.
7. *Mathematics as Related to Other Great Fields of Knowledge.* George Wolff.
8. *Form and Appreciation.* Griffith C. Evans.

The yearbook is published by the Bureau of Publications, Teachers College, Columbia University.

The seventy-seventh fascicule of the *Memorial des Sciences Mathématiques* is entitled *Théorie générale des réseaux, applications* and is the work of M. Cl. Guichard, professor of higher geometry at the Sorbonne. These fascicules are fairly comprehensive over-views of portions of mathematical fields and are of real value to the research worker in mathematics. Each number contains a list of titles and authors of all previous numbers. They are published by Gauthier-Villars, Quai des Grands-Augustins, 55, Paris.

The Econometric Society and the American Statistical Association will convene jointly in Chicago on the morning of December 28, 1936. The following papers will be presented:

1. *The Evolution of Fundamental Statistical Techniques.* Harold Hotelling, President of the Econometric Society.
2. *Some New Indexes of Agricultural Supplies and Carry-over.* E. J. Working, University of Illinois.

3. *New Indexes of Stock Prices and Yields and Their Relation to the Theories of Capital and Savings.* Charles F. Roos, Cowles Commission for Research in Economics.

The Annual Meeting of the American Mathematical Society will be held at Duke University, Durham, North Carolina, and the University of North Carolina, Chapel Hill, on December 29, 1936-January 1, 1937. As usual, the Mathematical Association of America will meet at the same time and places. Duke University has offered its dormitories on the men's campus to visiting professors and their guests free of charge.

Acta Mathematica is a well-known mathematical journal published in Uppsala by the firm of Almquist and Wiksells. It is devoted entirely to mathematical research, usually containing some five or six articles of a total length of one hundred fifty to two hundred pages. This journal was founded by G. Mittag-Leffler and is edited by N. E. Nörlund and T. Carleman. The subscription fee is thirty Swedish crowns per volume, post-free. The complete set—volumes 1-66 can be purchased for 1465 Swedish crowns. Papers intended for publication should be sent to N. E. Nörlund, University of Copenhagen, Denmark, or T. Carleman, Djursholm, Sweden.

Book Reviews

Edited by

P. K. SMITH

Plane and Spherical Trigonometry. By Jabir Shibli. Ginn and Company. New edition, 1936. xii+242 pages of text and 94 pages of tables.

In teaching trigonometry to students who are going to study analytic geometry and the calculus and who expect to use their knowledge of mathematics as engineers, it is well to keep a balance between analytic trigonometry and computation. In this book the author has alternated topics in trigonometric analysis with numerical solution of triangles and has thus secured balance and variety and avoided overdoing memorization of formulas and proving identities.

In the new edition some further explanations, exercises, and applications have been added, none of the material of the first edition being displaced by new material. The law of tangents and that of the functions of half-angles together with applications of solving oblique triangles and problems have been rearranged into a new chapter. The chapters on complex numbers, right spherical triangles, and oblique spherical triangles remain the same. The answers have been so much filled in that nearly all exercises and problems have answers.

The new edition maintains besides balance and variety of material simplicity, clearness, interest, and stimulation to independent thinking as did the first edition.

Vanderbilt University. WILSON L. MISER.

Analytic Geometry. By P. H. Graham, F. W. John, and H. R. Cooley. Prentice-Hall, New York, 1936. xx+294 pages.

It is becoming quite customary for authors of elementary mathematics textbooks to insert paragraphs about the history of the principles treated. This is a fortunate trend. The fact that the conic sections were given their present names over two millenniums ago by Apollonius, who studied them so thoroughly as to discover many of their important properties, is much more interesting to the student when he first meets these interesting curves than it will be at any

later date, especially if when he does come across this information it is buried among hundreds of similar facts and names in a book on the history of mathematics. By acquiring his history in this natural manner he is made to feel a kinship with the men and women who, through the ages, have contributed their bit to our present structure of mathematical theory. It is as though the student were himself synthesizing the contributions of Appollonius and Descartes. This volume contains about six pages of historical material. We wish it were thirty-six.

The historical paragraphs are not all that there is to recommend the book. It gives an excellent development of the material usually covered in a course of fifty to sixty recitations in analytic geometry. The explanations are detailed, the illustrative examples well selected and thoroughly explained, and the proofs are complete and rigorous. The treatment of the general second degree curve by means of its invariants is excellent and simple. The discussion of tangents, normals, subtangents, and subnormals is given in connection with each type of conic, not left to a later chapter. The general definition of a conic is introduced by means of a chapter on the intersection of a plane with a right circular cone.

The greatest deficiency of the book lies in the number and selection of the exercises. Those included are mostly of the drill type and are well suited to the mediocre student. There are probably enough exercises of this type for the average class, but very few which require sufficient original thought to tax the ingenuity of the more apt students of the class. Among other possible objections to the book are the small amount of attention given to equations in polar coordinates and the lack of an adequate treatment of intersections of curves in general.

The general appearance of the book is excellent and the type is large, clear, and free from serious errors. The figures deserve special mention, being numerous, well executed, and well selected.

College of St. Thomas. L. E. BUSH.

Graphical Solutions. By Charles O. Mackey. John Wiley and Sons, New York, 1936. 130+vii pages.

To reduce the amount of laborious calculating required in every engineering office, graphical solutions are extensively employed. The use of a few charts is included in most of the recent texts in mechanics, hydrodynamics, thermodynamics, applied electricity, etc. Their use

is desirable whenever the accuracy of the results obtained from them is commensurate with the accuracy of the data upon which the calculations are based.

Most engineers have received no formal instruction in the principles underlying the construction of charts for graphical solutions. These men usually test the correctness of a chart by using it to solve one or two problems of which they know the answers. If correct results are obtained, the chart is accepted.

Professor Mackey has succeeded in writing a text that should be of service to practicing engineers. It is readable and serves to take the mystery out of graphical solutions. He confines his illustrations to solutions that are of practical value to a large number of engineers. The numerous problems reach into all the principal engineering fields.

The book includes stationary adjacent scales, sliding scales, network charts, alignment charts, and the making of empirical equations from experimental data.

The text was written for use in the author's class which is elective for juniors and seniors; the course consists of two recitations a week for fifteen weeks. The author states, "The book is not a treatise on graphical computations. I have made no attempt to read and abstract everything that has been written on the subject. Instead, the text contains the material believed to be sufficient for a well-rounded course. The treatment is elementary and the mathematics simple. By omitting those few problems in which the calculus is used, the course might very properly be offered to freshmen." I agree with Professor Mackey but I should hesitate to try to teach the text to first year students.

Virginia Military Institute. S. W. ANDERSON.

The Mathematical Theory of Finance. By Kenneth P. Williams. The Macmillan Company, New York, 1935. xii+280 pages. $2.75.

Chapter I treats of simple interest and simple discount. The author's idea of simple interest, being illogical, seems unfounded from the very definition of simple interest.

Chapter II presents ideas concerning compound interest, together with formulas for finding the amount and present value. A comparison is made of compound interest and simple interest for fractional parts of conversion periods, with emphasis placed on the compound interest. The reader gets the idea that compound interest for fractional parts of conversion periods is used by the commercial world.

Nothing is mentioned about discounting amounts for a mixed number of conversion periods.

Chapter III introduces adequately the equation of value and the equation of time.

Chapter IV contains ideas on annuities with simple and compound interest, nominal rate, present value of an annuity, annuity schedules, annuity due, deferred annuities. The quantity n represents the number of payments instead of the length of the term. There are no problems involving present values of annuities between payments.

Chapter V develops general annuities by finding the interest rate per conversion period and then uses ordinary annuity formulas developed in the preceding chapter. This is well presented and should be easy for the students to grasp. As some of the problems read, it is not always clear what the term of the annuity is because the quantity n represents different things for annuities.

Chapter VI presents sinking funds and amortization, the annuity that will amount to 1, the annuity that 1 will buy, the relation between $s_{\overline{n}|}$ and $a_{\overline{n}|}$, and the amortization of a bonded debt. Each idea is illustrated with an example.

Bonds are analyzed in the next chapter by employing the investment rate, price to produce a given-yield, calculation of premiums, bonds bought between interest dates, bond tables, amortization of premiums, accumulation of the discount, expendable income and the yield of a bond. The yield is found from the expendable income and the purchase price, by interpolating in bond tables and by successive approximations. The text is not clear as to whom the earned interest, between interest dates, belongs.

Chapter VIII contains ideas concerning depreciation and replacement which are built around the straight line method of depreciation, the sinking fund method, the constant percentage method, the composite life of a plant, perpetuities, capitalized cost and evaluation of mining properties.

The next chapter presents life annuities, mortality tables, the American experience and the American annuitants' tables, pure endowments, deferred life annuities, temporary life annuities, annuities with death benefits, auxiliary theorems and applications of probabilities. These topics are well explained.

The last chapter treats of life insurance, built around whole life policies, paid up insurance, term insurance, endowment insurance, reserves on policies, surrender of policies and dividends.

Chapter X is followed by a list of supplementary exercises for each chapter. This list together with the problems in each chapter

offer plenty of thought work and adequate opportunity to apply the principles presented.

The appendix contains short reviews of the binomial theorem, arithmetic and geometric progression and logarithms.

At the end of the book you find the following tables: the number of each day of the year, ordinary and exact simple interest on 1000 at 1%, amount and the logarithm of the amount of 1 at compound interest, present value and the logarithm of the present value of 1 at compound interest, amount and the logarithm of the amount of 1 per annum, present value and the logarithm of the present value of 1 per annum, annuity which 1 will buy, amount and logarithm of the amount of 1 for fractional periods, nominal rate and logarithm of nominal rate of interest j converted p times corresponding to the effective rate i, amount and logarithm of amount at end of year of p deposits each of $1/p$ deposited at end of each pth part of a year, American experience table of mortality, American Annuitants' Table of mortality, and six place logarithms.

Answers are given to some of the problems.

One of the nice features of the book is the set of logarithm tables for the various quantities used. These tables will enable the student to do a great deal in a little space.

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